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GEOMETRY OF A SIMULTANEOUS SYSTEM OF TWO LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS OF THE SECOND ORDER*

BY

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In a former paper† I have laid the foundation for a general theory of invariants of a system of linear homogeneous differential equations. I have actually determined the invariants for the special case of two equations, each of the second order, i. e., for the system

$$\begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z &= 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z &= 0, \end{aligned} \tag{1}$$

the independent variable being x. The transformations, which were considered, were the most general which could convert (1) into another system of the same form and order, viz: the transformations of the group G

$$x = f(\xi), \quad y = \alpha(\xi)\eta + \beta(\xi)\zeta, \quad z = \gamma(\xi)\eta + \delta(\xi)\zeta, \tag{2}$$

where f, α, β, γ, δ are arbitrary functions of ξ, subject only to the condition that

$$\alpha\delta - \beta\gamma$$

must not vanish identically.

The present paper, besides deducing some new theorems, will be mainly concerned with geometrical interpretations. We shall again confine ourselves to the special case of equations (1) for two reasons. In the first place this will enable us to make use of the concrete results of our former paper, and in the second place we can thus avoid the consideration of configurations in hyperspace. It will not be difficult to generalize our considerations so as to include the general case, if only a space of the proper number of dimensions be employed.

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† E. J. WILCZYNSKI, *Invariants of systems of linear differential equations*, Transactions of the American Mathematical Society, vol. 2 (1901), no. 1.

Geometrical considerations of a similar nature have been applied to the theory of a single linear differential equation by HALPHEN and FANO.*

§1. *Definition of the general solutions, and of a simultaneous fundamental system of solutions.*

According to the fundamental theorem of the theory of differential equations, the equations (1) define two functions of x, which are analytic if the coefficients are analytic, and which can be made to satisfy the further conditions that y, z, dy/dx, dz/dx assume arbitrarily prescribed values for the particular value of $x = x_0$, provided that the coefficients p_{ik}, q_{ik}, are holomorphic in the vicinity of $x = x_0$.

Such a system of functions, y and z, is said to constitute a system of general solutions of (1).

Now let (y_i, z_i) for $i = 1, 2, 3, 4$, be four systems of simultaneous solutions of the given system (1), so that

$$\begin{aligned} y_i'' + p_{11}y_i' + p_{12}z_i' + q_{11}y_i + q_{12}z_i &= 0, \\ z_i'' + p_{21}y_i' + p_{22}z_i' + q_{21}y_i + q_{22}z_i &= 0 \end{aligned} \qquad (i=1, 2, 3, 4). \tag{3}$$

Then, denoting by c_1, c_2, c_3, c_4 four arbitrary constants,

$$y = \sum_{i=1}^{4} c_i y_i, \qquad z = \sum_{i=1}^{4} c_i z_i, \tag{4}$$

will also form a simultaneous system of solutions. Moreover, they constitute a system of general solutions, if, and only if, the determinant

$$D = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ z_1 & z_2 & z_3 & z_4 \\ z_1' & z_2' & z_3' & z_4' \end{vmatrix} \tag{5}$$

does not vanish identically. For then, if x_0 is a value of x for which $D \neq 0$, $c_1 \cdots c_4$ can be so determined as to make y, z, dy/dx, dz/dx equal to arbitrarily assigned constants for $x = x_0$.

If (y_i, z_i) are such functions of x which satisfy (1) and do not make D identically zero, we say of them that they constitute a *fundamental system* of simultaneous solutions. Equations (4) then furnish the general integrals of system (1).

We can express the condition

$$D \neq 0$$

* For the history of this subject Mr. FANO's excellent paper in Mathematische Annalen, vol. 53, may be consulted.

in another way. If $D=0$, it is possible to find four functions of x, λ, μ, ν, ρ, so that the four equations

$$\lambda y_k + \mu y_k' + \nu z_k + \rho z_k' = 0 \qquad (k=1, 2, 3, 4), \tag{6}$$

may be verified.

If (y_k, z_k) form a fundamental system of simultaneous solutions, it must therefore be impossible to find functions λ, μ, ν, ρ so as to satisfy (6), or what amounts to the same thing, it must be impossible to find functions α, β, γ, δ which satisfy the system of equations

$$\begin{aligned} &ay_1 + \beta y_2 + \gamma y_3 + \delta y_4 = 0, \\ &ay_1' + \beta y_2' + \gamma y_3' + \delta y_4' = 0, \\ &az_1 + \beta z_2 + \gamma z_3 + \delta z_4 = 0, \\ &az_1' + \beta z_2' + \gamma z_3' + \delta z_4' = 0, \end{aligned} \tag{7}$$

which, in particular, prevents $y_1 \cdots y_4$ and $z_1 \cdots z_4$ from verifying simultaneous relations of the form

$$\sum_{i=1}^{4} c_i y_i = 0, \quad \sum_{i=1}^{4} c_i z_i = 0, \tag{8}$$

where $c_1 \cdots c_4$ are constants.

Suppose now that four pairs of functions (y_i, z_i) which verify no relations of the form (6) or (7) are given. Then we can consider them as constituting a fundamental system of simultaneous solutions of a system of differential equations of the form (1). The relations between the coefficients p_{ik}, q_{ik} and the functions (y_k, z_k) follow from (3). They are

$$\begin{aligned} &Dp_{11} = -D(y_k'', z_k', y_k, z_k), \quad Dp_{12} = -D(y_k', y_k'', y_k, z_k), \\ &Dp_{21} = -D(z_k'', z_k', y_k, z_k), \quad Dp_{22} = -D(y_k', z_k'', y_k, z_k), \\ &Dq_{11} = -D(y_k', z_k', y_k'', z_k), \quad Dq_{12} = -D(y_k', z_k', y_k, y_k''), \\ &Dq_{21} = -D(y_k', z_k', z_k'', z_k), \quad Dq_{22} = -D(y_k', z_k', y_k, z_k''), \\ &\qquad D = D(y_k', z_k', y_k, z_k), \end{aligned} \tag{9}$$

where we have adopted the notation

$$D(a_k, b_k, c_k, d_k) = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}. \tag{10}$$

We have moreover

$$p_{11} + p_{22} = -\frac{1}{D}\frac{dD}{dx}, \tag{11}$$

whence

$$D = Ce^{-\int (p_{11}+p_{22})dx}, \tag{12}$$

where C is a non-vanishing constant.

All of these theorems are simple generalizations of well-known facts, but it is necessary for the purposes of this paper to formulate them explicitly.

If we subject the general solutions of our system (1) to a transformation of the form

$$\eta = ay + \beta z, \quad \zeta = \gamma y + \delta z, \tag{13}$$

where a, β, γ, δ are functions of x, then η and ζ will be the general solutions of a system of differential equations of the same form as (1). Moreover, if we put

$$\begin{aligned} \eta_i &= ay_i + \beta z_i, \\ \zeta_i &= \gamma y_i + \delta z_i \end{aligned} \qquad (i = 1, 2, 3, 4), \tag{13a}$$

the four pairs of functions (η_i, ζ_i) will form a definite fundamental system of the new system of differential equations, and its general solutions will be

$$\eta = \sum_{i=1}^{4} c_i \eta_i, \quad \zeta = \sum_{i=1}^{4} c_i \zeta_i.$$

Thus, if we consider instead of a pair of general solutions of system (1), four pairs of solutions which form a fundamental system, these are transformed cogrediently with each other, and with the pair of general solutions.

§ 2. *Geometrical interpretation.*

Let us interpret (y_1, y_2, y_3, y_4) and (z_1, z_2, z_3, z_4) as the homogeneous coördinates of two points in space. If (1) is integrated, we shall have all of these quantities expressed as functions of x:

$$y_k = f_k(x), \quad z_k = g_k(x) \qquad (k = 1, 2, 3, 4), \tag{14}$$

so that we can say that the system (1) defines two curves in space C_y and C_z, whose points moreover are put into a definite correspondence with one another, namely, those being corresponding points that belong to the same value of x.

But there is a restriction on these curves, owing to the condition that (y_i, z_i) are to be the members of a fundamental system, so that equations (6) must not be verified. Let us write (6) as follows

$$\lambda y_k + \mu y_k' = -(\nu z_k + \rho z_k') \qquad (k = 1, 2, 3, 4). \tag{6a}$$

Now the quantities $\lambda y_k + \mu y_k'$ for each value of x denote the homogeneous coordinates of some point on the tangent to the curve C_y constructed at the point whose parameter is x, and $\nu z_k + \rho z_k'$ denote the coördinates of a point on the

tangent to C_z at the corresponding point of this curve. If equation $(6a)$ is satisfied these two tangents intersect for all pairs of corresponding points.

In order, then, that the curves C_y and C_z may be the integral curves of a system of form (1), *it is necessary and sufficient that the tangents of these curves at all corresponding points do not intersect.*

In particular the curves may be plane curves, but they must not be in the same plane.

What is the geometrical meaning of transformations of form (13)?

Let us mark on the curves C_y and C_z the points P_y and P_z corresponding to the same value of x, and let us join them by a straight line L_{yz}. Then it is clear that the transformations (13), or more properly $(13a)$, convert the points P_y and P_z of the line L_{yz} into two other points P_η and P_ζ of the same line. Moreover, since $\alpha, \beta, \gamma, \delta$ are arbitrary functions of x, it becomes possible to convert P_y and P_z into *any* other two points of this line. The line L_{yz} then is invariant for all such transformations.

If we make this construction for all values of x, we obtain a ruled surface S, whose generators are the straight lines L_{yz}. A change of the independent variable

$$x = f(\xi),$$

where $f(\xi)$ is an arbitrary function, interchanges the generators in the most general way.

Thus, there belongs to every system of two linear homogeneous differential equations of the second order a ruled surface, which we shall call the integrating ruled surface, whose generators are the lines joining corresponding points of the two integral curves. This ruled surface is the same for all such systems which can be transformed into each other by a transformation of the form

$$\begin{aligned} \eta &= \alpha y + \beta z, \\ \zeta &= \gamma y + \delta z, \end{aligned} \qquad \xi = f(x), \tag{15}$$

where $\alpha, \beta, \gamma, \delta, f$ are arbitrary functions of x.

There is one important restriction, however, viz.: *the ruled surface cannot be a developable surface.* For, in the case of a developable surface, and only in that case, would the corresponding tangents of the two integral curves be coplanar, and therefore intersect.

This consideration also teaches us the meaning of the singular values of the independent variable for which

$$D = 0.$$

They are the values of x for which two consecutive generators of the ruled surface intersect, i. e., they correspond to the generators upon which are situated singular points of the surface.

It should be remarked that the ruled surface S will be different for different fundamental systems of solutions of (1), if we regard the tetrahedron of reference as fixed. All of the ruled surfaces belonging to the different possible fundamental systems of the same system of differential equations are obtained from one of them by projective transformation.

By means of equations (13a) we associated with each fundamental system of the original system of equations, a definite fundamental system of the transformed system of equations, and it is only for such associated fundamental systems of the two systems of equations that it is true that they have the same integrating ruled surface. In general the two integrating ruled surfaces will only be projective transformations of each other. If we call two systems of differential equations of form (1) *equivalent*, when they can be transformed into each other by a transformation of the form (15), we can state our theorem more precisely as follows:

If two systems of differentiel equations of form (1) *are equivalent, their integrating ruled surfaces are projective transformations of each other. Moreover if the fundamental systems of solutions be properly selected, the ruled surfaces coincide.*

Conversely, if the ruled surfaces of two such systems coincide, the systems are equivalent.

If the ruled surface is not of the second order, this converse is clear at once. For the arbitrary functions $\alpha, \beta, \gamma, \delta$ in the transformation (13) can be chosen so as to convert any pair of curves on the surface, which are not generating straight lines, into any other pair, for instance the pair of curves corresponding to the first system into that corresponding to the second. Moreover this ensures the equivalence of the two systems, if the surface is not of the second order. But if it is of the second order it may be generated by straight lines in two different ways. But even then the theorem holds, since we can transform one set of generators into the other by a projective transformation.

Suppose that any non-developable ruled surface S is given. There corresponds to it a class of mutually equivalent systems of linear differential equations. To find a representative of this class, we trace any two curves, not generating straight lines, on the surface and express the coördinates of their points as functions of a parameter x, in such a way that to the same value of x correspond points of the two curves which are situated upon the same generator of the surface. The system of differential equations whose integral curves are these, parametrically represented in this manner, is the required representative system.

We see further that it is always possible, by a transformation of form (15) to convert any system of differential equations of the kind here considered into another with plane integral curves. We can even take the planes of these curves parallel, and moreover this can be done in an infinity of ways.

If the ruled surface is algebraic, the system can be transformed into one whose integral curves are algebraic space or plane curves.

Any equation or system of equations between p_{ik}, q_{ik}, *etc., which remains invariant for all of the transformations of the form* (15), *expresses a projective property of the integrating ruled surface.* For, it expresses a property common to all possible pairs of curves on this surface, and p_{ik}, q_{ik}, p'_{ik}, $\cdots$ are differential invariants of the general projective group of space. *Conversely any projective property of the integrating ruled surface can be expressed by an invariant system of equations.*

§ 3. *Expression of the integrating ruled surface in line coördinates.*

The surface S is generated by the motion of the line L_{yz} which joins the points $y = (y_1, y_2, y_3, y_4)$ of the curve C_y and $z = (z_1, z_2, z_3, z_4)$ of the curve C_z.

The Plückerian line-coördinates of L_{yz} are the six determinants of the second order in the matrix

$$\begin{vmatrix} y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \end{vmatrix}.$$

Put

$$\omega_{ik} = y_i z_k - y_k z_i \qquad (i, k = 1, 2, 3, 4), \tag{16}$$

then the identical relation will hold

$$\omega_{12}\omega_{34} + \omega_{23}\omega_{14} + \omega_{31}\omega_{24} = 0. \tag{17}$$

Corresponding to the transformations of the infinite group G, we shall have for ω_{ik} the transformations

$$\omega_{ik} = (a\delta - \beta\gamma)\bar{\omega}_{ik}, \quad x = f(\xi).$$

But the six line-coördinates will verify a linear homogeneous differential equation, in general of the 6th order, say

$$P_0\omega^{(6)} + P_1\omega^{(5)} + P_2\omega^{(4)} + P_3\omega^{(3)} + P_4\omega^{(2)} + P_5\omega^{(1)} + P_6\omega = 0. \tag{18}$$

Now the most general transformations which convert such an equation into another of the same form and order are those of the infinite group H,

$$\omega = \phi(x)\bar{\omega}, \quad x = f(\xi),$$

where f and ϕ are arbitrary functions. Such functions of $P_1, \cdots, P_6$, $P'_1, \cdots, P'_6$, etc., which remain unaltered by all transformations of the group H are called invariants of (18).

It is clear, then, that the invariants of the linear differential equation of the sixth order, which the line-coördinates of a generator of the integrating ruled

surface satisfy, are also invariants of the original system of differential equations.

We proceed to set up this linear differential equation of the sixth order in a normal form. We have shown, in a former paper, that every system of form (1) can be reduced to a particular form, which we have called the *semi-canonical* form, for which $p_{ik} = 0$.* Let (y, z) and (η, ζ) be any two simultaneous solutions of a system of differential equations in the semi-canonical form, so that we shall have

$$\begin{aligned} y'' &= -q_{11}y - q_{12}z, & \eta'' &= -q_{11}\eta - q_{12}\zeta, \\ z'' &= -q_{21}y - q_{22}z, & \zeta'' &= -q_{21}\eta - q_{22}\zeta. \end{aligned} \tag{19}$$

Let us put

$$\omega = y\zeta - z\eta = (y\zeta). \tag{20}$$

We wish to find the differential equation satisfied by ω. To do this, let us first notice the relations

$$(\eta y'') = +q_{12}\omega, \quad (\eta z'') = +q_{22}\omega, \quad (\zeta y'') = -q_{11}\omega, \quad (\zeta z'') = -q_{21}\omega, \tag{21}$$

which are obtained from (19). Then, by successive differentiation, and using the abbreviations

$$\begin{aligned} 2v &= \omega'' + (q_{11} + q_{22})\omega, \\ w &= v'' - 2(q_{11}q_{22} - q_{12}q_{21})\omega + (q_{11} + q_{22})v, \\ t &= w' - (q_{11}q'_{22} + q_{22}q'_{11} - q_{12}q'_{21} - q_{21}q'_{12})\omega + (q'_{11} + q'_{22})v, \\ \rho &= t' - (q_{11}q''_{22} + q_{22}q''_{11} - q_{12}q''_{21} - q_{21}q''_{12})\omega + (q''_{11} + q''_{22})v, \end{aligned} \tag{22}$$

we obtain the system of equations

$$\begin{aligned} \omega' &= -(\eta z') + (\zeta y'), \\ v' &= q_{11}(\eta z') + q_{12}(\zeta z') - q_{21}(\eta y') - q_{22}(\zeta y'), \\ w &= q'_{11}(\eta z') + q'_{12}(\zeta z') - q'_{21}(\eta y') - q'_{22}(\zeta y'), \\ t &= q''_{11}(\eta z') + q''_{12}(\zeta z') - q''_{21}(\eta y') - q''_{22}(\zeta y'), \\ \rho &= q^{(3)}_{11}(\eta z') + q^{(3)}_{12}(\zeta z') - q^{(3)}_{21}(\eta y') - q^{(3)}_{22}(\zeta y'). \end{aligned} \tag{23}$$

Eliminating the determinants $(\eta z')$, etc., we find the required differential equation of the 6th order for ω, viz:

$$\begin{vmatrix} \rho, & q^{(3)}_{11}, & q^{(3)}_{12}, & -q^{(3)}_{21}, & -q^{(3)}_{22} \\ t, & q''_{11}, & q''_{12}, & -q''_{21}, & -q''_{22} \\ w, & q'_{11}, & q'_{12}, & -q'_{21}, & -q'_{22} \\ v', & q_{11}, & q_{12}, & -q_{21}, & -q_{22} \\ \omega', & -1, & 0, & 0, & +1 \end{vmatrix} = 0. \tag{24}$$

*Transactions of the American Mathematical Society, vol. 2 (1901), no. 1, p. 19.

This differential equation reduces to the 5th order if the minor of ρ in this determinant is zero, i. e., if

$$\begin{vmatrix} q_{11}''' - q_{22}''', & q_{12}''', & q_{21}''' \\ q_{11}'' - q_{22}'', & q_{12}'', & q_{21}'' \\ q_{11}' - q_{22}', & q_{12}', & q_{21}' \end{vmatrix} = 0,$$

or, when the equation is not written in the semi-canonical form, if

$$\Delta = \begin{vmatrix} u_{11} - u_{22}, & u_{12}, & u_{21} \\ v_{11} - v_{22}, & v_{12}, & v_{21} \\ w_{11} - w_{22}, & w_{12}, & w_{21} \end{vmatrix} = 0. \tag{25}$$

It is clear from this, that Δ must be an invariant, obviously of weight nine. This may also be verified analytically. Δ, being isobaric of weight 9, cannot be rationally expressed as a function of the seminvariants involved in it, for it can be shown that no such rational invariant of weight 9 exists. But there is a simple syzygy between Δ and the invariant of weight 18.* By expressing both of these quantities in terms of u_{ik}, v_{ik}, w_{ik}, it will be found that

$$\Delta^2 + 4\theta_{18} = 0. \tag{26}$$

We have seen that the differential equation for ω reduces to the 5th order if and only if $\Delta = 0$.

Therefore, the equation

$$\Delta = 0 \tag{27}$$

means, that the generators of the integrating ruled surface belong to a linear complex; for, the six line coördinates ω_{ik} must, in this case, satisfy a homogeneous linear relation with constant coefficients. The condition $\Delta = 0$ is necessary and sufficient.

If the linear complex, to which the generators of the ruled surface belong, is special, additional relations must be fulfilled. For any two linear complexes which are not special can be transformed into each other by a projective transformation, while a non-special complex can never be so transformed into a special complex.

A special complex consists of all lines which intersect a given straight line, called the axis of the complex. If

$$\omega_{12}a_{34} + \omega_{23}a_{14} + \omega_{31}a_{24} + \omega_{14}a_{23} + \omega_{24}a_{31} + \omega_{34}a_{12} = 0$$

is its equation, then a_{ik} are the line-coördinates of its axis, and

$$a_{12}a_{34} + a_{23}a_{14} + a_{31}a_{24} = 0.$$

*Transactions of the American Mathematical Society, vol. 2, no. 1, p. 18.

Every ruled surface contained in such a complex, therefore has upon it a straight line which is not a generator of the surface, namely the axis of the complex. Take this as our fundamental curve C_y, and also as one of the edges of the tetrahedron of reference.

Then we can put

$$y_3 = y_4 = 0 \tag{28}$$

while $y_1, y_2; z_1, z_2, z_3, z_4$ may be arbitrary functions of x. Since $y_3 = y_4 = 0$ we can take as the first differential equation of our system the linear differential equation of the second order of which y_1, y_2 form a fundamental system, so that

$$p_{12} = q_{12} = 0,$$

whence follows

$$u_{12} = v_{12} = w_{12} = 0, \tag{29}$$

which makes

$$\theta_{10} = \theta_{15} = \theta_{18} = 0, \tag{30}$$

while θ_4 and θ_6 do not vanish, unless there is a further specialization. Since, moreover, in this case

$$\theta_4 = (u_{11} - u_{22})^2,$$

we may assume

$$u_{11} - u_{22} \neq 0. \tag{29a}$$

The conditions (29) and (29a) are sufficient to insure that the integrating ruled surface shall be generated by lines belonging to a special linear complex, for from them follows $p_{12} = q_{12} = 0$. But they are not necessary. The necessary and sufficient conditions must form an invariant system of equations, which the conditions here found do not do. The conditions (30) on the other hand are necessary, as they form an invariant system, but they are not sufficient.

To find conditions for this case, which are both necessary and sufficient we proceed as follows. If the ruled surface belongs to a special linear complex, it must be possible to transform the given system, by a transformation of the form

$$y = a_{11}\eta + a_{12}\zeta,$$

$$z = a_{21}\eta + a_{22}\zeta,$$

into another, for which $\pi_{12} = \rho_{12} = 0$, if the coefficients of the transformed system be denoted by Greek letters.

But the conditions $\pi_{12} = \rho_{12} = 0$, give

$$2(a'_{12}a_{22} - a'_{22}a_{12}) + p_{11}a_{12}a_{22} + p_{12}a_{22}^2 - p_{21}a_{12}^2 - p_{22}a_{22}a_{12} = 0, \tag{31}$$

and

$$\begin{aligned} a''_{12}a_{22} - a''_{22}a_{12} + p_{11}a'_{12}a_{22} + p_{12}a'_{22}a_{22} - p_{21}a'_{12}a_{12} - p_{22}a'_{22}a_{12} \\ + q_{11}a_{12}a_{22} + q_{12}a_{22}^2 - q_{21}a_{12}^2 - q_{22}a_{22}a_{12} = 0. \end{aligned} \tag{32}$$

Differentiating the left member of (31) and subtracting from the resulting expression twice the left member of (32), we find

$$(33)\quad \begin{aligned}&(a_{12}a'_{22} - a'_{12}a_{22})(p_{11} + p_{22}) + a_{12}a_{22}[p'_{11} - 2q_{11} - (p'_{22} - 2q_{22})]\\ &\qquad + a_{22}^2(p'_{12} - 2q_{12}) - a_{12}^2(p'_{21} - 2q_{21}) = 0.\end{aligned}$$

Multiplying (31) by $-(p_{11} + p_{22})$ and (33) by 2, and adding we find

$$(34)\quad a_{12}a_{22}(u_{11} - u_{22}) + a_{22}^2u_{12} - a_{12}^2u_{21} = 0,$$

where the quantities u_{ik} are the same as those so denoted in our previous paper.

Dividing (34) by a_{22}^2, we have

$$\frac{a_{12}}{a_{22}}(u_{11} - u_{22}) + u_{12} - \left(\frac{a_{12}}{a_{22}}\right)^2 u_{21} = 0,$$

whence by differentiation

$$(35)\quad \left(u_{11} - u_{22} - 2\frac{a_{12}}{a_{22}}u_{21}\right)\frac{d}{dx}\frac{a_{12}}{a_{22}} = -u'_{12} - \frac{a_{12}}{a_{22}}(u'_{11} - u'_{22}) + \left(\frac{a_{12}}{a_{22}}\right)^2 u'_{21}.$$

But from (31) we have

$$2\frac{d}{dx}\frac{a_{12}}{a_{22}} = -\frac{a_{12}}{a_{22}}(p_{11} - p_{22}) - p_{12} + \left(\frac{a_{12}}{a_{22}}\right)^2 p_{21},$$

which gives on substitution in (35), after simplifying and clearing of fractions,

$$(36)\quad a_{12}a_{22}(v_{11} - v_{22}) + a_{22}^2v_{12} - a_{12}^2v_{21} = 0.$$

Differentiating (36), we get by a repetition of this process

$$(37)\quad a_{12}a_{22}(w_{11} - w_{22}) + a_{22}^2w_{12} - a_{12}^2w_{21} = 0.$$

We have then three equations (34), (36) and (37) for

$$a = \frac{a_{12}}{a_{22}}.$$

It is at once seen that Δ must vanish, and if we put

$$(38)\quad \begin{aligned}&2(v_{12}w_{21} - v_{21}w_{12}) = U, \quad 2(w_{12}u_{21} - w_{21}u_{12}) = V, \quad 2(u_{12}v_{21} - u_{21}v_{12}) = W,\\ &vw_{12} - wv_{12} = U_{12}, \qquad wu_{12} - uw_{12} = V_{12}, \qquad uv_{12} - vu_{12} = W_{12},\\ &vw_{21} - wv_{21} = U_{21}, \qquad wu_{21} - uw_{21} = V_{21}, \qquad uv_{21} - vu_{21} = W_{21},\\ &u_{11} - u_{22} = u, \quad v_{11} - v_{22} = v, \quad w_{11} - w_{22} = w,\end{aligned}$$

we obtain

$$(39)\quad \begin{aligned}a &= -\frac{U}{2U_{21}} = -\frac{V}{2V_{21}} = -\frac{W}{2W_{21}},\\ a^2 &= \frac{U_{12}}{U_{21}} = \frac{V_{12}}{V_{21}} = \frac{W_{12}}{W_{21}},\end{aligned}$$

whence

$$\begin{aligned} &VW_{21} - WV_{21} = WU_{21} - UW_{21} = UV_{21} - VU_{21} = 0, \\ (40)\quad &V_{12}W_{21} - W_{12}V_{21} = W_{12}U_{21} - U_{12}W_{21} = U_{12}V_{21} - V_{12}U_{21} = 0, \\ &U^2 - 4U_{12}U_{21} = V^2 - 4V_{12}V_{21} = W^2 - 4W_{12}W_{21} = 0. \end{aligned}$$

It will be noted that U, V, W, etc., are the nine minors of the second order of the determinant Δ, and that *all of the minors of the second order of their determinant*

$$(41)\qquad \begin{vmatrix} U & U_{12} & U_{21} \\ V & V_{12} & V_{21} \\ W & W_{12} & W_{21} \end{vmatrix}$$

vanish in consequence of the relations (40). Moreover the quantities U, U_{12}, U_{21}; V, V_{12}, V_{21}; W, W_{12}, W_{21}; are all cogredient with each other and u, u_{12}, u_{21}, etc., for transformations of the dependent variables, and we can write

$$(42)\qquad \theta_{10} = \tfrac{1}{4}W^2 - W_{12}W_{21}.$$

The conditions (40) which are necessary for the case under consideration are also sufficient. For, let us suppose these conditions fulfilled for a system of differential equations. Then let us transform this system as before, putting now

$$(43)\qquad a_{12} = -\tfrac{1}{2}W, \quad a_{22} = W_{21}.$$

We find by actual computation that the left member of (31), which is equal to π_{12}, vanishes. Denoting this left member, in general, by π_{12}, and the left member of (32) by ρ_{12}, the left member of (34) will be

$$2\pi'_{12} - 4\rho_{12} - (p_{11} + p_{22})\pi_{12}.$$

But this vanishes under the given conditions, and since $\pi_{12} = 0$, ρ_{12} also will vanish. Therefore the first equation of the transformed system will actually have the form

$$(44)\qquad \frac{d^2\eta}{dx^2} + \pi_{11}\frac{d\eta}{dx} + \rho_{11}\eta = 0,$$

which proves that the ruled surface in this case belongs to a special linear complex. This proof is insufficient only if $W = W_{21} = 0$.

If $W = W_{21} = 0$, we have either $u_{21} = v_{21} = 0$, or else also $W_{12} = 0$. In the first case, if $u_{21} = v_{21} = 0$, and $u_{11} - u_{22} \neq 0$ we find from the equations defining these quantities that p_{21} and q_{21} must vanish, which means that in this case no transformation is necessary, one of the original equations already being of form (44). If however, in addition to $u_{21} = v_{21} = 0$, we have $u = 0$, we find from (40) that either $W_{12} = 0$, which comes under our second case, or else U, V, U_{21} and

V_{21} must all vanish. But this again gives either $u = 0$, $u_{12} = 0$, $u_{21} = 0$, which we shall see later makes the ruled surface of the second order, or else $w_{21} = 0$. This again gives rise to two sub-cases. Either $p_{21} = q_{21} = 0$ as in a former case, or $v = 0$ and therefore $u_{12} = 0$, which is again the case of a surface of the second order. In all of these cases the ruled surface belongs to a special linear complex.

In the second case mentioned above, we have $W = W_{12} = W_{21} = 0$, or

$$u_{12}v_{21} - u_{21}v_{12} = 0, \quad uv_{12} - vu_{12} = 0, \quad uv_{21} - vu_{21} = 0,$$

whence, denoting by ρ a proportionality factor,

$$v = \rho u, \quad v_{12} = \rho u_{12}, \quad v_{21} = \rho u_{21}, \tag{45}$$

unless $u = u_{12} = u_{21} = 0$, which is the case of a surface of the second degree.

Using the equations (32) of our former paper,* which defined v_{ik} we find from (45)

$$\begin{aligned} \rho u &= 2u' + 2(p_{12}u_{21} - p_{21}u_{12}), \\ \rho u_{12} &= 2u'_{12} + (p_{11} - p_{22})u_{12} - p_{12}u, \\ \rho u_{21} &= 2u'_{21} - (p_{11} - p_{22})u_{21} + p_{21}u, \end{aligned}$$

and similarly, from the equations (39) in our former paper,

$$w = (\rho^2 + 2\rho')u, \quad w_{12} = (\rho^2 + 2\rho')u_{12}, \quad w_{21} = (\rho^2 + 2\rho')u_{21}, \tag{46}$$

but these expressions of v_{ik}, w_{ik} show that, in this case, not only

$$W = W_{12} = W_{21} = 0,$$

but also

$$U = U_{12} = U_{21} = V = V_{12} = V_{21} = 0,$$

i. e., all of the minors of the second order of Δ vanish. In this case also, the ruled surface belongs to a special linear complex although our method of reduction fails. That this is so will, however, be apparent from what follows.

We have seen, then, that the equations (40) *are the necessary and sufficient conditions for an integrating ruled surface belonging to a special linear complex.* Moreover we have seen how the axis of this complex, i. e., the straight line on the surface, which is not one of the generators, may be found, for this is the geometrical meaning of the transformation which we have just completed. The cases in which this transformation fails are those in which more than one such straight line exists on the surface.

If the generators of the ruled surface belong to two different linear complexes,

$$\begin{aligned} \Omega_1 &= \omega_{12}a_{34} + \omega_{23}a_{14} + \omega_{31}a_{24} + \omega_{14}a_{23} + \omega_{24}a_{31} + \omega_{34}a_{12} = 0, \\ \Omega_2 &= \omega_{12}b_{34} + \omega_{23}b_{14} + \omega_{31}b_{24} + \omega_{14}b_{23} + \omega_{24}b_{31} + \omega_{34}b_{12} = 0, \end{aligned} \tag{47}$$

*Transactions of the American Mathematical Society, vol. 2, no. 1, p. 8.

or if they are lines of what PLÜCKER calls a linear congruence, they belong to every one of the ∞^1 complexes

$$\Omega_1 + \lambda\Omega_2 = 0. \tag{48}$$

Now, among these there are in general two, which are special. For (48) represents a special linear complex, if and only if,

$$B\lambda^2 + C\lambda + A = 0, \tag{49}$$

where we have put

$$\begin{aligned} A &= a_{12}a_{34} + a_{23}a_{14} + a_{31}a_{24}, \\ B &= b_{12}b_{34} + b_{23}b_{14} + b_{31}b_{24}, \\ C &= a_{12}b_{34} + a_{23}b_{14} + a_{31}b_{24} + a_{34}b_{12} + a_{14}b_{23} + a_{24}b_{31}. \end{aligned} \tag{50}$$

But (49) has two distinct and determinate roots if

$$C^2 - 4AB \neq 0.$$

In this case we can assume that the complexes (47) are special, so that $A = B = 0$, and consequently $C \neq 0$. The congruence consists of all of the lines intersecting two distinct, non-intersecting straight lines, the directrices of the congruence.

If $C^2 - 4AB = 0$, we have a congruence with coincident directrices.

A sub-case of this,

$$A = B = C = 0,$$

must be excluded from our investigation. For in this case the directrices of the congruence intersect, so that the congruence consists of all straight lines intersecting a pair of intersecting straight lines. In other words, all of the lines of the congruence are situated either in the plane of the directrices, or else pass through their point of intersection. Any ruled surface contained in such a congruence is either a plane or a cone, and therefore a developable surface. But a developable surface cannot be the integrating ruled surface of our system of differential equations.

It will be easily seen that the necessary and sufficient conditions, which must be fulfilled in order that the generators of the integrating ruled surface may belong to a linear congruence, are that all of the minors of the second order of Δ *must vanish.*

In this case then

$$U = U_{12} = U_{21} = V = V_{12} = V_{21} = W = W_{12} = W_{21} = 0, \tag{51}$$

and as a consequence,

$$\theta_{10} = \theta_{15} = \theta_{18} = 0.$$

In the case of a congruence with coincident directrices additional conditions must be fulfilled. Such a congruence may be described in a number of ways. It consists of all of the lines of a complex which intersect a given line of the complex. Or, we may consider the line with which the two coincident directrices of the congruence coincide, as a generator of a surface of the second order. All of the tangents to this surface, passing through the given generator, constitute the lines of the congruence. Finally, for actual use, we can define such a congruence in the following manner. Take rectangular axes x, y, z in space. Through a point P on the x axis, and the origin O pass a plane, whose projection on the yz plane makes an angle λ with the z axis, where

$$\tan \lambda = \frac{k}{x},$$

k being a constant, and x the distance of P from the origin. Then the lines of this plane, which pass through P, are the lines of the congruence.*

Let us take the x axis, which must be a line on our surface, intersected by all of the generators, as the fundamental curve C_y, and the abscissa $OP = x$ of the point P as the independent variable of our system of differential equations. Moreover, suppose that we have chosen our homogeneous coördinates so that the ratios

$$\frac{y_1}{y_4}, \quad \frac{y_2}{y_4}, \quad \frac{y_3}{y_4}; \qquad \frac{z_1}{z_4}, \quad \frac{z_2}{z_4}, \quad \frac{z_3}{z_4};$$

are the cartesian coördinates of the points (y) and (z) respectively. Then we can put

$$y_1 = x, \quad y_2 = 0, \quad y_3 = 0, \quad y_4 = 1. \tag{52}$$

Moreover the equation of the plane mentioned above, will be

$$y = z \tan \lambda,$$

whence follows the single relation between z_2 and z_3,

$$z_2 = z_3 \tan \lambda = z_3 \frac{k}{x},$$

or

$$xz_2 - kz_3 = 0. \tag{53}$$

There exists then, in this case, a system of differential equations, equivalent to the given system, for which

$$\begin{aligned} &y_1 = x, \quad y_2 = 0, \quad y_3 = 0, \quad y_4 = 1, \\ &z_1 = z_1, \quad z_2 = z_2, \quad z_3 = \frac{x}{k} z_2, \quad z_4 = 1, \end{aligned} \tag{54}$$

* PLÜCKER, *Neue Geometrie des Raumes*, pp. 57 and 73.

where z_1 and z_2 are arbitrary functions of x.

Substituting these values in equations (9), we find

$$
\begin{aligned}
&p_{11} = p_{12} = q_{11} = q_{12} = 0, \\
&p_{21} = -\frac{1}{z_2^2}\left[z_1'' z_2^2 - 2z_2 z_1' z_2' - (x - z_1)\{2(z_2')^2 - z_2 z_2''\}\right], \\
&p_{22} = -2\frac{z_2'}{z_2}, \\
&q_{21} = -\frac{1}{z_2^2}\left[2(z_2')^2 - z_2 z_2''\right], \quad q_{22} = +\frac{1}{z_2^2}\left[2(z_2')^2 - z_2 z_2''\right],
\end{aligned}
\tag{55}
$$

whence we derive the relations

$$
\begin{aligned}
&2p_{22}' - 4q_{22} + p_{22}^2 = 0, \\
&q_{21} + q_{22} = 0, \\
&p_{11} = p_{12} = q_{11} = q_{12} = 0.
\end{aligned}
\tag{56}
$$

Therefore

$$
u_{11} = u_{12} = u_{22} = 0, \quad v_{11} = v_{12} = v_{22} = 0, \quad w_{11} = w_{12} = w_{22} = 0. \tag{57}
$$

Substituting these values, it will be seen that *all of the invariants vanish.*

The conditions (57) are sufficient to make the ruled surface belong to a congruence with coincident directrices. For, if we assume in the first place the additional relation $u_{21} = 0$, the surface will be of the second order, as we shall show very soon, and therefore it will be a surface of the kind here considered. If $u_{21} \neq 0$, we find from the conditions $u_{11} = u_{12} = v_{11} = 0$, that p_{12} must vanish. From $u_{12} = 0$ then follows $q_{12} = 0$, and from $u_{11} = u_{22} = 0$,

$$
2p_{11}' - 4q_{11} + p_{11}^2 = 2p_{22}' - 4q_{22} + p_{22}^2 = 0. \tag{58}
$$

Our system therefore has the form

$$
\begin{aligned}
&y'' + p_{11}y' + q_{11}y = 0, \\
&z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0,
\end{aligned}
$$

with the relations (58). If we now transform, by putting

$$
y = e^{-\frac{1}{2}\int p_{11}dx}\eta, \quad z = e^{-\frac{1}{2}\int p_{22}dx}\zeta,
$$

the system will become, owing to the relations (58),

$$
\begin{aligned}
&\eta'' = 0, \\
&\zeta'' + \pi\eta' + \kappa\eta = 0,
\end{aligned}
$$

where π and κ are functions of x subject to no condition.

But this system has the fundamental system of solutions

$$\eta_1 = x, \qquad \eta_2 = 0, \quad \eta_3 = 0, \quad \eta_4 = 1,$$
$$\zeta_1 = -\int\int(\pi + \kappa x)\,dx^2, \quad \zeta_2 = k, \quad \zeta_3 = x, \quad \zeta_4 = -\int\int \kappa dx^2,$$

where k is a constant, which must be different from zero, for we find that

$$D(\eta_k', \zeta_k', \eta_k, \zeta_k) = k.$$

Making another transformation, by putting

$$\bar{\zeta} = \frac{\zeta}{-\int\int \kappa dx^2},$$

it will be seen that $\eta_1, \eta_2, \eta_3, \eta_4$ and $\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3, \bar{\zeta}_4$ form a fundamental system of solutions of a system equivalent to the given system, and of the form (54). This proves that any system of differential equations, for which the relations (57) are satisfied, has a ruled surface which belongs to a linear congruence with coincident directrices.

But the sufficient conditions (57) are not necessary. The following conditions

$$I^2 - 4J = 0, \quad K - I'^2 = 0, \quad L - 4I''^2 = 0, \tag{59}$$

are both necessary and sufficient. For, in the first place it may be verified that they form an invariant system of equations.

In the second place, any system of differential equations of the kind here considered can be so transformed that the relations (57) shall be satisfied. But then the relations (59) will also be satisfied, and moreover, since these relations form an invariant system of equations, they must have been satisfied for the untransformed system of differential equations. The equations (59) are then necessary conditions. But they are also sufficient. For, suppose that they are satisfied. We can then assume

$$u_{11} - u_{22} = 0, \quad u_{12}u_{21} = 0.$$

For, we can, by a transformation of the form $y = \lambda\eta$, $z = \mu\zeta$, make u_{11} and u_{22} equal to any functions of x whatever, and therefore it may be so chosen as to make $u_{11} = 0$ and $u_{22} = 0$.

Moreover, since then $u_{12}u_{21} = 0$, either u_{12} or u_{21} must vanish. Suppose $u_{21} \neq 0$; then $u_{12} = 0$, and $v_{12} = 0$, while

$$v_{11} - v_{22} = 2p_{12}u_{21}$$

must vanish, on account of (59). But since $u_{21} \neq 0$, we have $p_{12} = 0$, whence $v_{11} = v_{22} = 0$, and also $w_{11} = w_{22} = w_{12} = 0$, or, taken together, the sufficient conditions (57). If $u_{21} = 0$, together with $u_{11} = u_{22} = u_{12} = 0$, we have the case of a ruled surface of the second order, which also belongs to a linear congruence with

coincident directrices. Had we assumed $u_{12} \neq 0$, instead of $u_{21} \neq 0$, we should have found the conditions

$$u_{11} = u_{21} = u_{22} = 0, \quad v_{11} = v_{21} = v_{22} = 0, \quad w_{11} = w_{21} = w_{22} = 0,$$

which are the same as (57) except that the order of the two differential equations is interchanged, i. e., except for the notation.

The necessary and sufficient conditions for a ruled surface belonging to a linear congruence with coincident directrices, are therefore

$$\begin{aligned} (u_{11} - u_{22})^2 + 4u_{12}u_{21} &= 0, \\ (v_{11} - v_{22})^2 + 4v_{12}v_{21} &= 0, \\ (w_{11} - w_{22})^2 + 4w_{12}w_{21} &= 0, \end{aligned} \tag{60}$$

for these are equivalent to (59).

If the ruled surface is made up of the lines common to three distinct linear complexes, it is a surface of the second order (not a cone), and all of the elements of the determinant Δ must vanish.

Therefore the necessary and sufficient conditions for an integrating ruled surface of the second order are

$$u_{11} - u_{22} = u_{12} = u_{21} = 0. \tag{61}$$

This is the case treated in our former paper in § 4. It is possible to so transform the system that the new dependent variables η and ζ satisfy the same linear differential equation of the second order

$$\eta'' + p\eta' + q\eta = 0;$$

the geometrical sense of this reduction is that any two generators of the second kind are taken as the fundamental curves C_y and C_z. The differential equations for y and z are then the same because the point-rows cut out of any two generators of the second kind by a moving generator of the surface are projective.

This reduction can be made analytically by reducing to the semi-canonical form.

§ 4. *Geometrical interpretation of the semi-canonical form of the system of differential equations.*

We have shown in our former paper, that every system of differential equations of the form here considered, can be reduced to the form

$$\begin{aligned} y'' + \rho_{11}y + \rho_{12}z &= 0, \\ z'' + \rho_{21}y + \rho_{22}z &= 0, \end{aligned} \tag{62}$$

which we have called the semi-canonical form. We have also determined the

most general subgroup of the group (2), which leaves this semi-canonical form invariant. Its finite equations are,

$$(63) \qquad \xi = \xi(x), \quad \eta = \sqrt{\frac{d\xi}{dx}}\,(C_{11}y + C_{12}z), \quad \zeta = \sqrt{\frac{d\xi}{dx}}\,(C_{21}y + C_{22}z),$$

where C_{ik} are constants.

Now, of course, equations (62) define a pair of curves on the integrating ruled surface. This is transformed by equations (63) into another pair, whose defining system of differential equations also has the semi-canonical form.

Since (63) is the most general transformation which leaves the semi-canonical form invariant, *we see that there exists upon the integrating ruled surface a single infinity of curves, whose defining systems of differential equations have the semi-canonical form. It is also clear from equations* (63) *that a moving generator of the ruled surface intersects any four of these curves in a row of four points whose anharmonic ratio is constant.*

On any ruled surface the generating straight lines constitute one set of asymptotic lines. *We shall now prove that the curves just mentioned constitute the other set of asymptotic lines.*

Let the homogeneous coördinates be so chosen that

$$\frac{y_1}{y_4}, \frac{y_2}{y_4}, \frac{y_3}{y_4}; \quad \frac{z_1}{z_4}, \frac{z_2}{z_4}, \frac{z_3}{z_4};$$

are the cartesian coördinates of the points describing the curves C_y and C_z respectively. Then if X, Y, Z denote the coördinates of any point in the osculating plane of the curve C_y at the point (y_1, y_2, y_3, y_4), the equation of this plane is

$$\begin{vmatrix} X - \frac{y_1}{y_4}, & Y - \frac{y_2}{y_4}, & Z - \frac{y_3}{y_4} \\ \frac{d}{dx}\frac{y_1}{y_4}, & \frac{d}{dx}\frac{y_2}{y_4}, & \frac{d}{dx}\frac{y_3}{y_4} \\ \frac{d^2}{dx^2}\frac{y_1}{y_4}, & \frac{d^2}{dx^2}\frac{y_2}{y_4}, & \frac{d^2}{dx^2}\frac{y_3}{y_4} \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} Xy_4 - y_1, & Yy_4 - y_2, & Zy_4 - y_3 \\ y_1y_4' - y_4y_1', & y_2y_4' - y_4y_2', & y_3y_4' - y_4y_3' \\ y_1y_4'' - y_4y_1'', & y_2y_4'' - y_4y_2'', & y_3y_4'' - y_4y_3'' \end{vmatrix} = 0.$$

But since y_i, z_i are solutions of (62), we have

$$\begin{aligned} y_i'' &= -\rho_{11}y_i - \rho_{12}z_i, \\ z_i'' &= -\rho_{21}y_i - \rho_{22}z_i \end{aligned} \qquad (i = 1, 2, 3, 4),$$

and, therefore,

$$y_i y_k'' - y_k y_i'' = -\rho_{12}(z_i y_k - y_i z_k),$$

which shows that the osculating plane of C_y at the point (y_1, y_2, y_3, y_4) also passes through (z_1, z_2, z_3, z_4), i. e., it coincides with the plane tangent to the surface at the point (y_1, y_2, y_3, y_4). This proves that C_y is an asymptotic curve of the surface.

Our remarks furnish a new proof of PAUL SERRET'S theorem: *the double ratio of the four points, in which any generator intersects four fixed asymptotic curves of the second kind, is constant.*

We shall leave the geometrical interpretation of the canonical form for a future occasion, when we shall also treat some other important problems not touched upon in this paper. In particular, it will be interesting to introduce the idea of duality into this theory.

UNIVERSITY OF CALIFORNIA,
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THEORY OF LINEAR GROUPS IN AN ARBITRARY FIELD*

BY

LEONARD EUGENE DICKSON

§ 1. *Introduction.*

Various branches of group theory may be correlated by a treatment of groups of transformations in a given field or domain of rationality. In view of the simplicity of their treatment and of their importance as well in applications as in the general theory,† groups of linear transformations offer a natural starting place in the construction of a theory of groups in a given domain of rationality.

The chief result of the present paper is the exhibition of four infinite systems of groups of transformations which are simple groups in every domain of rationality. For the case of the field of all complex numbers these groups are the simple continuous groups of LIE. By the well known investigations of KILLING and CARTAN, the latter groups give the only systems of simple continuous groups of a finite number of parameters.

As in the theories of algebraic and differential equations, so also in the theory of groups of transformations, it is of first importance that the definitions, conceptions and developments shall have reference to a given field or domain of rationality. For example, it is important to have a theory of continuous groups in the field of complex numbers and a theory in the field of real numbers. Two real continuous groups may not be isomorphic, although the corresponding complex groups are isomorphic.‡ If we allow complex transformations to canonical types, there results a complete list of real groups; allowing only reductions by real transformations, the list is often more extensive.

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† For example, a continuous group G with a finite number of parameters is simply or multiply isomorphic with a linear homogeneous group called the adjoint group of G. If G be simple, it may be exhibited as a linear fractional (projective) group. From the fundamental rôle played by the adjoint group and by the simple groups, the theory of linear groups is of capital importance in LIE's theory of continuous groups.

‡ DICKSON: Bulletin of the American Mathematical Society, (2) vol. 7 (1901), p. 340; SLOCUM: Proceedings of the American Academy of Arts and Sciences, vol. 36 (1900), p. 105.

The chief results in the theory of linear groups in a finite field are presented in the author's treatise on Linear Groups, * to which reference will here be made by the initials L. G.

In certain questions cencerning continuous groups, as that of the structure of a mixed group, the methods here presented often give additional knowledge, not obtained by following LIE's method.

In §§ 9–10 is investigated a group in an arbitrary field which corresponds to the simple continuous group of 14 parameters, an isolated group not in the four systems of LIE. For the case of a finite field of order p^n, we are led to a new simple group of order $p^{6n}(p^{6n}-1)(p^{2n}-1)$.

§ 2. *Definition of fields and groups.*

A set of operators forms a *group* if the following properties hold:

(a) The product (compound) of any two operators of the set is itself an operator of the set.

(b) The composition of operators is associative: if A, B, C are any operators of the set, then $(AB)C = A(BC)$.

(c) To every operator A of the set corresponds an operator A_1 of the set such that $AA_1 = A_1A = I$, where I is the operator identity, which leaves unaltered all possible operands. This A_1 is called the *inverse* of A and is designated A^{-1}.

A set of elements forms a *field* † if they can be combined by addition, subtraction, multiplication and division, the divisor not being the element zero (necessarily in the set), these operations being subject to laws of elementary algebra, and if the resulting sum, difference, product or quotient be uniquely determined as an element of the set.

A field may therefore be characterized by the property that *the rational operations of algebra can be performed within the field.*

As examples of fields may be noted the finite fields, ‡ the field R of all rational numbers, the field $R(i)$ of numbers $a + bi$, where a and b are rational, the field of all real numbers, the field C of all complex numbers, the field $R(\theta)$ of all rational functions of the algebraic number θ, a root of an equation belonging to and irreducible in the field R.

§ 3. *General linear homogeneous and linear fractional groups.*

Let $\xi_1, \xi_2, \cdots, \xi_m$ be arbitrary variables. Consider the linear homogeneous transformation

$$A: \qquad \xi_i' = \sum_{j=1}^{m} a_{ij}\xi_j \qquad (i = 1, 2, \cdots, m)$$

* B. G. Teubner, Leipsic, 1901.

† Domain of rationality or Körper. See WEBER's *Algebra*.

‡ Each is necessarily a Galois field of order a power of a prime (MOORE).

with coefficients a_{ij} in a given field F such that the determinant

$$|A| \equiv |a_{ij}| \neq 0 \qquad (i, j = 1, 2, \cdots, m).$$

Such a transformation A will be said to belong to the field F. Consider a second transformation belonging to F,

$$A': \qquad \xi'_i = \sum_{j=1}^{m} a'_{ij} \xi_j \qquad (i = 1, 2, \cdots, m).$$

By the compound, or product, of A and A' we mean the transformation

$$A'': \qquad \xi'_i = \sum_{j=1}^{m} a''_{ij} \xi_j \qquad (i = 1, 2, \cdots, m),$$

where

$$a''_{ij} \equiv \sum_{k=1}^{m} a'_{ik} a_{kj} \qquad (i, j = 1, 2, \cdots, m).$$

We have here compounded the transformations in the order A, A'; the relation between A, A', A'' is written

$$AA' = A''.$$

By the theorem for the multiplication of determinants,

$$|A''| \equiv |a''_{ij}| = |a'_{ik}| \cdot |a_{kj}| = |A'| \cdot |A| \neq 0.$$

Since a_{ij} and a'_{ij} belong to the field F by hypothesis, the coefficients a''_{ij} also belong to F. Hence the product A'' is a transformation belonging to the field F. The transformation

$$A^{-1}: \qquad \xi'_i = \sum_{j=1}^{m} \frac{A_{ji}}{|A|} \xi_j \qquad (i = 1, 2, \cdots, m),$$

where A_{ji} is the adjoint (first minor with proper sign) of a_{ji} in $|a_{ji}|$, has its coefficients in the field F and has the determinant $|A|^{-1}$. The product AA^{-1} is the identity I; indeed, it replaces ξ_i by the function *

$$\sum_{j,k}^{1,\cdots,m} \frac{A_{ji} a_{jk}}{|A|} \xi_k = \sum_{k=1}^{m} \frac{\delta_{ik} |A|}{|A|} \xi_k = \xi_i.$$

Hence the inverse A^{-1} of A is a transformation in the field F. Moreover, the transformations of the form A are seen to obey the associative law [§ 2, property (b)]. It follows that the totality of transformations A constitutes a group. It will be called *the general linear homogeneous group on m variables with coefficients in the field F* and denoted by the symbol † $GLH(m, F)$.

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* In KRONECKER's notation, $\delta_{ii} = 1$, $\delta_{ik} = 0 (k \neq i)$.

† A finite field is uniquely defined by its order, necessarily a power of a prime number, p^n (MOORE). The corresponding group is $GLH(m, p^n)$. [See L. G., §§ 97–98.]

The group $GLH(m, F)$ is generated by the transformations

$$D_{1,\lambda}: \quad \xi_1' = \lambda\xi_1, \quad \xi_i' = \xi_i \qquad (i = 2, 3, \cdots, m),$$

$$B_{r,s,\lambda}: \quad \xi_r' = \xi_r + \lambda\xi_s, \quad \xi_i' = \xi_i \qquad (i = 1, \cdots, m;\ i \neq r, r \neq s),$$

where λ is an arbitrary quantity $\neq 0$ in the field F.

The proof is identical with that in L. G., § 100. The proof shows that any transformation A of the group can be expressed uniquely as a product $A_1 D_{1,\lambda}$, where $\lambda = |A|$ and where A_1 is derived from the transformations $B_{r,s,k}$, all of which have determinant unity.

The totality of transformations A of determinant unity forms a group Γ called *the special linear homogeneous group $SLH(m, F)$*. It is generated by the transformations $B_{r,s,\lambda}$.

The product $B^{-1}AB$ is called *the transform of A by B*. Since

$$|B^{-1}AB| = |B^{-1}| \cdot |A| \cdot |B| = |B|^{-1} \cdot |A| \cdot |B| = |A|,$$

the transform of A has the same determinant as A.

A subgroup Γ of G is called *invariant (self-conjugate) under G* if the transform of each transformation of Γ by an arbitrary transformation of G belongs to $[\Gamma]$, i. e., symbolically, if $g^{-1}\gamma g = \gamma'$.

The group $SLH(m, F)$ is an invariant subgroup of $GLH(m, F)$.

By making A_1 correspond to $A \equiv A_1 D_{1,|A|}$, we establish an isomorphism of $SLH(m, F)$ with $GLH(m, F)$. The identity corresponds to the (commutative) group of the transformations $D_{1,\lambda}$. The latter is called the *quotient-group* of the general by the special linear group.

The special linear homogeneous group Γ contains an invariant subgroup H composed of the transformations

$$M_\mu: \quad \xi_i' = \mu\xi_i \qquad [\mu^m = 1] \qquad (i = 1, \cdots, m).$$

Let J be an invariant subgroup of Γ which contains all the transformations M_μ and still other transformations. By the proof in L. G., § 104, interpreted for an infinite field F, it follows that $J \equiv \Gamma$ if $m > 2$; while, by § 105, it follows, for $m = 2$, that J contains a transformation of the form $B_{2,1,\lambda\rho^2}$, with $\lambda \neq 0$, and ρ an arbitrary quantity in the field F. Having $B_{2,1,\lambda\sigma^2}$, the *group* J contains its inverse $B_{2,1,-\lambda\sigma^2}$. Hence J contains the product, in which ρ and σ are arbitrary in F,

$$B_{2,1,\lambda\rho^2} B_{2,1,-\lambda\sigma^2} \equiv B_{2,1,\lambda(\rho^2-\sigma^2)}.$$

To make $\rho^2 - \sigma^2 = \tau$, an arbitrary quantity in F, we set

$$\rho = \tfrac{1}{2}(\tau + 1), \quad \sigma = \tfrac{1}{2}(\tau - 1),$$

thereby excluding the case in which F has a modulus $p = 2$.

With $B_{2,1,\lambda\tau}$, J contains every $B_{2,1,\kappa}$, κ arbitrary in F. But $B_{2,1,\kappa}$ is transformed into $B_{1,2,-\kappa}$ by $(\xi_1' = \xi_2,\ \xi_2' = -\xi_1)$, which belongs to Γ. Since $B_{1,2,\kappa}$ and $B_{2,1,\kappa}$ generate Γ, we have $J = \Gamma$. Hence, for any m, H is a maximal invariant subgroup of Γ. The quotient-group Γ/H is therefore *simple.*

The group $SLH(m, F)$ has $(f, 1)$ isomorphism with a simple group, where f is the number of solutions in F of $x^m = 1$, and F is any infinite field or any finite field of order p^n, provided $p^n > 3$ if $m = 2$.*

Introducing the linear fractional transformations

$$x_i' = \frac{a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im-1}x_{m-1} + a_{im}}{a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mm-1}x_{m-1} + a_{mm}} \qquad (i = 1, \cdots, m-1),$$

we may derive, as in L. G., § 108, the following theorem:

The group $LF(m, F)$ of all linear fractional transformations on $m-1$ variables with coefficients in an infinite field F and of determinant unity is a simple group.

§ 4. *The Abelian linear group.*

A linear homogeneous transformation on $2m$ variables with coefficients in a field F is called *Abelian* if, when operating simultaneously upon two sets of variables $\xi_i, \eta_i;\ \bar{\xi}_i, \bar{\eta}_i (i = 1, \cdots, m)$, it leaves formally invariant (up to a factor belonging to F) the function

$$\phi \equiv \sum_{i=1}^{m} (\xi_i \bar{\eta}_i - \eta_i \bar{\xi}_i).$$

The totality of such transformations constitutes the *general Abelian linear group* $GA(2m, F)$. Those transformations which leave ϕ absolutely invariant form a subgroup called the *special Abelian linear group* $SA(2m, F)$.

If F be a continuous field (real or complex), the group $SA(2m, F)$ is simple† (in LIE'S sense). If we take for ϕ the function

$$\sum_{i=1}^{m} (\xi_i d\eta_i - \eta_i d\xi_i),$$

we recognize $SA(2m, F)$ to be the homogeneous form of the largest projective group on $2m-1$ variables which leaves invariant a linear complex.‡

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* In speaking of the index of an invariant subgroup H of a group Γ of infinite order, we mean the number of right-hand multipliers M_i such that the products hM_i, when h runs through the set of operators of H, give once and but once every operator of Γ. If H is of order f, we say that Γ has $(f, 1)$ isomorphism with Γ/H.

† Bulletin of the American Mathematical Society (2), vol. 3 (1897), pp. 267–270. With LIE a simple group is one containing no invariant continuous subgroup.

‡ Ibid., pp. 270, 271.

THEOREM. *In any field F, the group $SA(2m, F)$ is generated by the transformations, all of determinant unity,*

$$M_i: \quad \xi_i' = \eta_i, \quad \eta_i' = -\xi_i;$$
$$L_{i,\lambda}: \quad \xi_i' = \xi_i + \lambda\eta_i;$$
$$N_{i,j,\lambda}: \quad \xi_i' = \xi_i + \lambda\eta_j, \quad \xi_j' = \xi_j' + \lambda\eta_i;$$

where λ is an arbitrary quantity in F. The group has a maximal invariant subgroup formed of the identity I and the transformation T which changes the signs of the $2m$ variables.‡ The case $m = 1$ is exceptional if F be of order 2 or 3.

The proof proceeds as in L. G., §§ 110, 111, 114, 116, the statements on p. 97, lines 1–3, being replaced by the following argument. Since J contains $L_{2,\lambda\tau^2}$, in which $\lambda \neq 0$ and τ is arbitrary in F, it contains the inverse $L_{2,-\lambda\tau^2}$ and therefore the product

$$L_{2,\lambda\tau_1^2} L_{2,-\lambda\tau_2^2} \equiv L_{2,\lambda(\tau_1^2-\tau_2^2)}.$$

Taking $\tau_1 = \frac{1}{2}(\kappa+1)$, $\tau_2 = \frac{1}{2}(\kappa-1)$, we reach $L_{2,\lambda\kappa}$, where κ is arbitrary in F. Hence J contains $L_{2,\mu}$, μ being arbitrary in F. An analogous change is to be made on p. 97, lines 25–28.

The group obtained as the quotient-group of $SA(2m, F)$ by $\{I, T\}$ will be designated by $A(2m, F)$. It is simple except in the cases $m = 1$, F of order 2 or 3.

THEOREM. *A transformation of period 2 of $SA(2m, F)$ is conjugate within that group with one of the m non-conjugate transformations*

$$T_{1,-1}, \quad T_{1,-1}T_{2,-1}, \quad T_{1,-1}T_{2,-1}T_{3,-1}, \quad \cdots, \quad T \equiv T_{1,-1}T_{2,-1}\cdots T_{m,-1},$$

where $T_{i,-1}$ alters only ξ_i and η_i whose signs it changes.

The proof proceeds as in L. G., §§ 120–121.

The study of the conjugacy of the operators of period 2 in the quotient-group $A(2m, F)$ is not so simple for infinite fields as for finite fields (L. G., §§ 122–123). For the simplest case $m = 1$, a transformation of period 4 in $SA(2, F)$ is conjugate within that group with one of the transformations

$$S_\gamma \equiv \begin{pmatrix} 0 & \gamma \\ -\gamma^{-1} & 0 \end{pmatrix}.$$

The most general transformation of determinant unity which transforms S_γ into S_δ has the form

$$\begin{pmatrix} d\delta\gamma^{-1}, & b\delta \\ -b\gamma^{-1}, & d \end{pmatrix} \qquad (\gamma/\delta = b^2 + d^2).$$

‡ For the second part of the theorem and for the remainder of this section, it is assumed that, *if* there be a modulus p for an infinite field F, $p \neq 2$.

Hence S_γ and S_δ are conjugate within $SA(2, F)$ if, and only if, the ratio $\gamma\,\delta$ is expressible as the sum of two squares (including zero) in the field F.

For the field of rational numbers, S_1 is conjugate with S_γ for $\gamma = 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, \cdots, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \cdots$, a series containing every prime number of the form $4t+1$, but no prime number of the form $4t+3$.* It follows readily that there is an infinite number of non-conjugate transformations of period 4, viz., $S_1, S_3, S_7, S_{21}, S_{11}, S_{33}, S_{77}, S_{231}, S_{19}, \cdots$, including every S_τ for which τ is a prime number of the form $4t+3$, of which there are an infinite number by DIRICHLET'S theorem.

In the field C of all complex numbers, every S_γ is conjugate with $S_1 \equiv M_1$ within the group. As in L. G., §§ 122–123, we obtain the theorem:

If $s \equiv m/2$ or $(m-1)/2$ according as m is even or odd, the group $A(2m, C)$ contains exactly $s+1$ sets of conjugate operators of period 2. As representatives we may take

$$M \equiv M_1 M_2 \cdots M_m,\ T_{1,-1},\ T_{1,-1}T_{2,-1}, \cdots, T_{1,-1}T_{2,-1}\cdots T_{s,-1}.$$

In the field of all real numbers every S_γ is conjugate within the group $SA(2, F)$ either with $S_1 \equiv M_1$ or else with $S_{-1} \equiv M_1 T_{1,-1}$. By a simple modification of the developments in L. G., § 123, we find that, within $SA(2m, F)$, every transformation S, such that $S^2 = T_{1,-1}T_{2,-1}\cdots T_{m,-1}$, is conjugate with one of the transformations M, $MT_{1,-1}$, $MT_{1,-1}T_{2,-1}$, $\cdots$, $MT_{1,-1}\cdots T_{m,-1}$. For $m=1$ these transformations have been shown to be not conjugate. That they are not conjugate when $m=2$ may be shown as follows. A transformation which transforms M_1M_2 into either $M_1M_2T_{1,-1}$ or $M_1M_2T_{1,-1}T_{2,-1}$ must have the form

$$\begin{Bmatrix} a_{11} & \gamma_{11} & a_{12} & \gamma_{12} \\ \gamma_{11} & -a_{11} & \gamma_{12} & -a_{12} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{Bmatrix},$$

so that, by one of the Abelian conditions,

$$-a_{11}^2 - \gamma_{11}^2 - a_{12}^2 - \gamma_{12}^2 = 1.$$

The commutative transformation $T_{1,-1}T_{2,-1}$ being introduced, it follows that

$$M_1M_2T_{1,-1}T_{2,-1}$$

is not conjugate with $M_1M_2T_{2,-1}$ or with M_1M_2.

For general m, it follows by a similar proof that neither M nor MT is conjugate with any one of the series $MT_{1,-1}$, $MT_{1,-1}T_{2,-1}$, $\cdots$, $MT_{1,-1}\cdots T_{m-1,-1}$ nor M with MT.

*Cf. WEBER, *Algebra*, vol. 1, 1st ed. (1895), p. 585.

For the field F of all real numbers, every operator of period two of the group $A(2m, F)$ is conjugate within the group with one of the following operators

$$T_{1,-1},\ T_{1,-1}T_{2,-1},\ \cdots,\ T_{1,-1}T_{2,-1}\cdots T_{s,-1},\ M \equiv M_1M_2\cdots M_m,$$

$$MT_{1,-1},\ MT_{1,-1}T_{2,-1},\ \cdots,\ MT_{1,-1}T_{s,-1}\cdots T_{3,-1},$$

where $s = m/2$ or $(m-1)/2$ according as m is even or odd.

At least in the cases $m = 1$ and $m = 2$, no two of these $2s+1$ operators are conjugate within the group.

§ 5. *A generalization of the Abelian linear group.*

Those linear homogeneous transformations on mq variables with coefficients in any field F which, if operating simultaneously upon q sets each of mq variables, the jth set being exhibited by the notation

$$x_{i1}^{(j)},\ x_{i2}^{(j)},\ \cdots,\ x_{iq}^{(j)} \qquad (i=1, 2, \cdots, m),$$

leave formally and absolutely invariant the function

$$\phi = \sum_{i=1}^{m} \begin{vmatrix} x_{i1}^{(1)}, & x_{i2}^{(1)}, & \cdots, & x_{iq}^{(1)} \\ \cdot & \cdot & \cdots & \cdot \\ x_{i1}^{(q)}, & x_{i2}^{(q)}, & \cdots, & x_{iq}^{(q)} \end{vmatrix},$$

form a group $G(m, q, F)$. For $q = 2$, it is the group $SA(m, F)$ of § 4.

Proceeding as in L. G., §§ 124–128, we obtain the theorems:*

For $q > 2$, the group $G(m, q, F)$ is generated by the substitutions

$$P_{ij} \equiv (x_{i1}x_{j1})(x_{i2}x_{j2})\cdots(x_{iq}x_{jq}) \qquad (i, j=1, 2, \cdots, m),$$

and the totality of transformations in F of determinant unity,

$$x'_{1l} = \sum_{k=1}^{q} a_{lk}x_{1k} \qquad (l=1, 2, \cdots, q).$$

For $q > 2$, $G(m, q, F)$ has an invariant subgroup which is the direct product of m commutative groups each the special linear homogeneous group

* For the case in which F is a continuous field, these theorems were established by the author (using the LIE theory) in the Bulletin of the American Mathematical Society (2), vol. 3, pp. 271–273, May, 1897.

in F on q variables, the quotient-group being generated by the substitutions P_{ij} and thus holoedrically isomorphic with the symmetric group on m letters.*

§ 6. *The compounds of a linear homogeneous group.*

To the transformation (a_{ij}) of § 3 we make correspond the transformation

$$[a]_q: \qquad Y'_{i_1, i_2, \cdots, i_q} = \sum_{l_1, \cdots, l_q} \begin{vmatrix} a_{i_1 l_1} & a_{i_1 l_2} \cdots a_{i_1 l_q} \\ a_{i_2 l_1} & a_{i_2 l_2} \cdots a_{i_2 l_q} \\ \cdots & \cdots \\ a_{i_q l_1} & a_{i_q l_2} \cdots a_{i_q l_q} \end{vmatrix} Y_{l_1, l_2, \cdots, l_q},$$

where the sets $i_1, i_2, \cdots, i_q$ and $l_1, l_2, \cdots, l_q$ take independently all the distinct combinations q at a time of the integers $1, 2, \cdots, m$. Likewise, to the transformation (a'_{ij}) we make correspond the transformation $[a']_q$. Then (L. G., § 153) to the product $(a''_{ij}) \equiv (a_{ij})(a'_{ij})$ corresponds $[a'']_q \equiv [a]_q [a']_q$. Hence, if the transformations (a_{ij}) form a group, the transformations $[a]_q$ form an isomorphic group called the qth *compound* of the former.

By simple modifications of §§ 154–164 of L. G., we obtain the results:

The general linear homogeneous group $GLH(m, F)$ has $(d, 1)$ isomorphism with its qth compound if the equation $x^q = 1$ has exactly d roots in F.

The special linear homogeneous group $SLH(m, F)$ has $(g, 1)$ isomorphism with its qth compound if the equations $x^q = 1$, $x^m = 1$ have in the field F exactly g common solutions.

The second compound of $GLH(m, F)$ leaves invariant the Pfaffian † $[1, 2, \cdots, m]$. *For m odd, the transformation $[a]_2$ gives rise to the transformation*

$$F'_i = \sum_{j=1}^{m} A_{ij} F_j \qquad (i = 1, 2, \cdots, m)$$

upon the Pfaffians $F_j \equiv [1, 2, \cdots, j-1, j+1, \cdots, m]$, where A_{ij} denotes the first minor of a_{ij} in the determinant $|a_{ij}|$. The transformation $[a]_2$ effects upon the $\frac{1}{2}m(m-1)$ Pfaffians $[i_1, i_2, \cdots, i_{m-2}]$, where each $i = 1, 2, \cdots, m$,

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* The transformations of the ith group are given by the formula

$$x'_{ij} = \sum_{k=1}^{q} \beta^{ij}_{ik} x_{ik}, \; x'_{sj} = x_{sj} \qquad (s = 1, \cdots, m;\ s \neq i;\ j = 1, \cdots, q),$$

where, for each $i \leqq m$, the determinant $|\beta^{ij}_{ik}| = 1 (j, k = 1, \cdots, q)$.

† Inversely, in a continuous field F, the largest linear group on $\frac{1}{2}m(m-1)$ variables which leaves the Pfaffian invariant is the second compound of $GLH(m, F)$, Bulletin of the American Mathematical Society, vol. 5 (1898), pp. 338–342.

such that $i_1 < i_2 < \cdots < i_{m-2}$, *a linear homogeneous transformation identical with the transformation* $[a]_{m-2}$ *of the* $(m-2)$*th compound.* *

The q*th and* $(m-q)$*th compounds of* $SLH(m, F)$ *are holoedrically isomorphic.* Indeed, the number of solutions in F of $x^m = 1$, $x^q = 1$ equals the number of solutions in F of $x^m = 1$, $x^{m-q} = 1$.

The general Abelian group $GA(2m, F)$ *is the largest linear homogeneous group in the field* F *on* $2m$ *variables whose second compound has the relative invariant*

$$Z \equiv Y_{12} + Y_{34} + \cdots + Y_{2m-1\,2m}.$$

The second compound of $SA(2m, F)$ *is a simple*† *group with the absolute invariants* Z *and the Pfaffian* $[1, 2 \cdots, 2m]$.

The simple group $A(4, F)$ *is holoedrically isomophic with a subgroup of the quinary linear group in* F *which leaves absolutely invariant*

$$Y^2 + Y_{13}Y_{24} - Y_{14}Y_{23}.$$

According as the field F *does not or does contain a primitive fourth root of unity, the second compound of* $SLH(4, F)$ *is holoedrically isomorphic with the simple group* $LF(4, F)$ *or has a maximal invariant subgroup* $\{I, T\}$, *where* T *changes the signs of the six variables, the quotient-group being holoedrically isomorphic with* $LF(4, F)$.

The second compound of $SLH(4, F)$ *contains the transformation*

$$Y'_{12} = \nu Y_{12}, \quad Y'_{13} = Y_{13}, \quad Y'_{14} = Y_{14}, \quad Y'_{23} = Y_{23}, \quad Y'_{24} = Y_{24}, \quad Y'_{34} = \nu^{-1} Y_{34}$$

if, and only if, ν *be a square in the field* F.

By § 3, the group $SLH(4, F)$ is generated by the transformations $B_{r,s,\lambda}$ $(r, s = 1, 2, 3, 4; r \neq s)$, and hence by $B_{1,2,\lambda}$ and

$$A_{12}: (\xi'_1 = \xi_2, \xi'_2 = -\xi_1), \quad A_{13}: (\xi'_1 = \xi_3, \xi'_3 = -\xi_1), \quad A_{34}: (\xi'_3 = \xi_4, \xi'_4 = -\xi_3).$$

The second compounds of these transformations are respectively

$$\begin{aligned}
B'_{1,2,\lambda}: &\quad Y'_{13} = Y_{13} + \lambda Y_{23}, \quad Y'_{14} = Y_{14} + \lambda Y_{24};\\
A'_{12}: &\quad Y'_{13} = Y_{23}, \quad Y'_{14} = Y_{24}, \quad Y'_{23} = -Y_{13}, \quad Y'_{24} = -Y_{14};\\
A'_{13}: &\quad Y'_{12} = -Y'_{23}, \quad Y'_{14} = Y_{34}, \quad Y'_{23} = Y_{12}, \quad Y'_{34} = -Y_{14};\\
A'_{34}: &\quad Y'_{13} = Y_{14}, \quad Y'_{14} = -Y_{13}, \quad Y'_{23} = Y_{24}, \quad Y'_{24} = -Y_{23}.
\end{aligned}$$

The second compound possesses the absolute invariant

$$\theta_6 \equiv Y_{12}Y_{34} - Y_{13}Y_{24} + Y_{14}Y_{23}.$$

* For the case of a continuous field F, these theorems were established by the author in the Bulletin of the American Mathematical Society, vol. 5 (1898), pp. 120–135.

† If there be a modulus p, we assume that $p \neq 2$; as also in the rest of the section.

The second compound of $SLH(4, F)$ may be exhibited as that senary linear group G_6, leaving $\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3$ invariant, which is generated by the transformations $S_{ij} \equiv (\xi_i\eta_i)(\xi_j\eta_j)$ and

$$W_{i,j,\lambda}: \qquad \xi_i' = \xi_i + \lambda\eta_j, \ \xi_j' = \xi_j - \lambda\eta_i.$$

The group G_6 will therefore contain the transformations

$$S_{ij}^{-1} W_{i,j,\lambda} S_{ij} \equiv V_{i,j,\lambda}: \qquad \eta_i' = \eta_i + \lambda\xi_j, \ \eta_j' = \eta_j - \lambda\xi_i;$$

$$S_{jk}^{-1} W_{i,j,\lambda} S_{jk} = Q_{i,j,\lambda}: \qquad \xi_i' = \xi_i + \lambda\xi_j, \ \eta_j' = \eta_j - \lambda\eta_i;$$

$$Q_{j,i,-1} Q_{i,j,1} Q_{j,i,-1} \equiv P_{ij} T_{j,-1};$$

where $P_{ij} \equiv (\xi_i\xi_j)(\eta_i\eta_j)$ and $T_{i,\lambda}$ denotes $\xi_i' = \lambda\xi_i$, $\eta_i' = \lambda^{-1}\eta_i$. Set

$$(1) \quad Y_{12} \equiv \xi_3, \ Y_{13} \equiv \xi_1, \ Y_{14} \equiv \xi_2, \ Y_{23} \equiv -\eta_2, \ Y_{24} \equiv \eta_1, \ Y_{34} \equiv -\eta_3.$$

Then $B_{1,2,\lambda}'$, A_{12}', A_{13}', A_{34}' become $W_{1,2,-\lambda}$, $S_{12}T_{2,-1}P_{12}$, $S_{23}P_{23}T_{2,-1}$, $P_{12}T_{2,-1}$, respectively. Hence G_6 contains the transformations which correspond to the generators of the second compound. Inversely, from them we derive $W_{1,2,\lambda}$, $P_{12}T_{2,-1}$, S_{12} and therefore the transform S_{13} of S_{12} by $S_{23}P_{23}T_{2,-1}$. We then derive $S_{12}S_{13} \equiv S_{23}$ and $P_{23}T_{2,-1}$. The latter transforms $W_{1,2,\lambda}$ into $W_{1,3,\lambda}$; and $P_{12}T_{2,-1}$ transforms $W_{1,3,-\lambda}$ into $W_{2,3,\lambda}$. We have therefore derived the generators S_{ij}, $W_{i,j,\lambda}$ of G_6.

By § 4 the special Abelian group $SA(4, F)$ is generated by M_1, M_2, $L_{1,\lambda}$, $L_{2,\lambda}$ and $N_{1,2,\lambda}$. Their second compounds are respectively A_{12}', A_{34}', $B_{1,2,\lambda}'$, $B_{3,4,\lambda}'$ and $N_{1,2,\lambda}'$, the first three being exhibited above,* while

$$B_{3,4,\lambda}': \qquad Y_{13}' = Y_{13} + \lambda Y_{14}, \quad Y_{23}' = Y_{23} + \lambda Y_{24};$$

$$N_{1,2,\lambda}': \qquad \begin{cases} Y_{12}' = Y_{12} - \lambda Y_{24}, & Y_{34}' = Y_{34} + \lambda Y_{24}, \\ Y_{13}' = Y_{13} + \lambda Y_{12} - \lambda Y_{34} - \lambda^2 Y_{24}. \end{cases}$$

The last five transformations leave invariant θ_6 and $Z \equiv Y_{12} + Y_{34}$, by the theorem stated above. We introduce the new variables †

$$(2) \quad \xi_0 = \tfrac{1}{2}(Y_{12} - Y_{34}), \quad \xi_1 = Y_{13}, \quad \eta_1 = Y_{24}, \quad \xi_2 = Y_{14}, \quad \eta_2 = -Y_{23}.$$

Then A_{12}', A_{34}', $B_{1,2,\lambda}'$ and $B_{3,4,\lambda}'$ become $S_{12}T_{2,-1}P_{12}$, $P_{12}T_{2,-1}$, $W_{1,2,-\lambda}$ and $Q_{1,2,\lambda}$ respectively. Finally, $N_{1,2,\lambda}'$ becomes $X_{0,1,\lambda}$, if we use the notation

$$X_{0,j,\lambda}: \qquad \xi_0' = \xi_0 - \lambda\eta_j, \quad \xi_j' = \xi_j + 2\lambda\xi_0 - \lambda^2\eta_j.$$

For later use, we introduce, for the transform of $X_{0,j,\lambda}$ by $S_{jk}(k \neq j)$,

$$Y_{0,j,\lambda}: \qquad \xi_0' = \xi_0 - \lambda\xi_j, \quad \eta_j' = \eta_j + 2\lambda\xi_0 - \lambda^2\xi_j.$$

* The present variables ξ_1, η_1, ξ_2, η_2 correspond to the former ξ_1, ξ_2, ξ_3, ξ_4 respectively.

† The last four have the same definition as in the case of the group G_6.

The second compound of $SA(4, F)$ is a simple group which may be exhibited as that quinary linear group G_5 leaving $\xi_0^2 + \xi_1\eta_1 + \xi_2\eta_2$ invariant which is generated by $P_{12}T_{2,-1}$, S_{12}, $W_{1,2,\lambda}$, $Q_{1,2,\lambda}$ and $X_{0,1,\lambda}$.

We obtain an important subgroup of G_5 as follows. The second compound of the special Abelian transformation

$$\begin{Bmatrix} \alpha & 0 & 0 & \beta \\ 0 & \delta & \gamma & 0 \\ 0 & \beta & \alpha & 0 \\ \gamma & 0 & 0 & \delta \end{Bmatrix} \qquad (\alpha\delta - \beta\gamma = 1)$$

affects only four of the six variables Y_{ij} and has the form

	Y_{12}	Y_{13}	Y_{24}	Y_{34}
$Y'_{12} =$	$\alpha\delta$	$\alpha\gamma$	$-\beta\delta$	$-\beta\gamma$
$Y'_{13} =$	$\alpha\beta$	α^2	$-\beta^2$	$-\alpha\beta$
$Y'_{24} =$	$-\gamma\delta$	$-\gamma^2$	δ^2	$\gamma\delta$
$Y'_{34} =$	$-\beta\gamma$	$-\alpha\gamma$	$\beta\delta$	$\alpha\delta$

The latter leaves $Y_{12} + Y_{34}$ invariant. Expressed in terms of the variables ξ_0, ξ_1, η_1 defined by (2), it takes the form

(3)

	ξ_0	ξ_1	η_1
$\xi'_0 =$	$\alpha\delta + \beta\gamma$	$\alpha\gamma$	$-\beta\delta$
$\xi'_1 =$	$2\alpha\beta$	α^2	$-\beta^2$
$\eta'_1 =$	$-2\gamma\delta$	$-\gamma^2$	δ^2

$(\alpha\delta - \beta\gamma = 1)$,

a transformation of determinant unity leaving $\xi_0^2 + \xi_1\eta_1$ absolutely invariant.* *The transformations* (3) *form a simple group isomorphic with* $LF(2, F)$.

To exhibit a subgroup of both G_5 and G_6 which leaves $\xi_1\eta_1 + \xi_2\eta_2$ invariant, we form the second compound of

$$\begin{Bmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & C & D \end{Bmatrix} \qquad \begin{pmatrix} \alpha\delta - \beta\gamma = 1 \\ AD - BC = 1 \end{pmatrix},$$

* Among them occur $X_{0,1,\lambda}$ and $Y_{0,1,\lambda}$, the latter for $\alpha = \delta = 1$, $\beta = 0$, $\gamma = -\lambda$.

and obtain a transformation affecting only Y_{13}, Y_{14}, Y_{23}, Y_{24}. Introducing ξ_1, ξ_2, $-\eta_2$, η_1, respectively, for the former [as in (1) or (2)], we obtain the transformation

$$(4)\qquad \begin{array}{c|cccc} & \xi_1 & \eta_1 & \xi_2 & \eta_2 \\ \hline \xi_1' = & \alpha A & \beta B & \alpha B & -\beta A \\ \eta_1' = & \gamma C & \delta D & \gamma D & -\delta C \\ \xi_2' = & \alpha C & \beta D & \alpha D & -\beta C \\ \eta_2' = & -\gamma A & -\delta B & -\gamma B & \delta A \end{array}$$

The group of these transformations has, in view of its origin, the factor groups $LF(2, F)$, $LF(2, F)$, and $\{I, T\}$, where T changes the sign of each variable.*

§ 7. *Concerning linear groups with quadratic invariants.*

Consider the group $G(m, F)$ of linear homogeneous transformations

$$S: \quad \begin{cases} \xi_i' = a_{i0}\xi_0 + \sum_{j=1}^{m} (a_{ij}\xi_j + \gamma_{ij}\eta_j) & (i = 0, 1, \cdots, m), \\ \eta_i' = \beta_{i0}\xi_0 + \sum_{j=1}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j) & (i = 1, \cdots, m), \end{cases}$$

with coefficients in a field† F, which have the absolute invariant

$$g_m \equiv \xi_0^2 + \xi_1\eta_1 + \xi_2\eta_2 + \cdots + \xi_m\eta_m.$$

The conditions for the formal invariance of g_m are

$$(5)\qquad a_{00}^2 + \sum_{i=1}^{m} a_{i0}\beta_{i0} = 1,$$

$$(6)\qquad 2a_{0j}a_{0k} + \sum_{i=1}^{m} (a_{ij}\beta_{ik} + a_{ik}\beta_{ij}) = 0 \qquad (j = 0, 1, \cdots, m;\ k = 1, \cdots, m),$$

$$(7)\qquad 2\gamma_{0j}\gamma_{0k} + \sum_{i=1}^{m} (\gamma_{ij}\delta_{ik} + \gamma_{ik}\delta_{ij}) = 0 \qquad (j, k = 1, \cdots, m),$$

$$(8)\qquad 2a_{0j}\gamma_{0k} + \sum_{i=1}^{m} (a_{ij}\delta_{ik} + \gamma_{ik}\beta_{ij}) = \begin{cases} 1 & (k = j), \\ 0 & (k \neq j), \end{cases} \qquad (j = 0, 1, \cdots, m;\ k = 1, \cdots, m).$$

* To compare with the earlier proof for finite fields, American Journal of Mathematics, vol. 21 (1899), p. 248, we have only to replace β by $-\gamma$, γ by $-\beta$, B by C, C by B.

† If there be a modulus p, we assume that $p \neq 2$. For a finite field of order 2^n, the structure was given by the writer in the American Journal of Mathematics, vol. 21 (1899), p. 243.

It follows from these relations that the inverse of S is

$$S^{-1}: \quad \begin{cases} \xi_0' = a_{00}\xi_0 + \frac{1}{2}\sum_{j=1}^{m}(\beta_{j0}\xi_j + a_{j0}\eta_j) \\ \xi_i' = 2\gamma_{0i}\xi_0 + \sum_{j=1}^{m}(\delta_{ji}\xi_j + \gamma_{ji}\eta_j) \qquad (i=1,\cdots,m). \\ \eta_i' = 2a_{0i}\xi_0 + \sum_{j=1}^{m}(\beta_{ji}\xi_j + a_{ji}\eta_j) \end{cases}$$

The conditions for the invariance of g_m under S_3^{-1} are seen to be

$$a_{00}^2 + 4\sum_{i=1}^{m} a_{0i}\gamma_{0i} = 1, \tag{9}$$

$$\tfrac{1}{2}a_{j0}a_{k0} + \sum_{i=1}^{m}(a_{ji}\gamma_{ki} + a_{ki}\gamma_{ji}) = 0 \tag{10}$$
$$(j=0,1,\cdots,m\,;\,k=1,\cdots,m),$$

$$\tfrac{1}{2}\beta_{j0}\beta_{k0} + \sum_{i=1}^{m}(\beta_{ji}\delta_{ki} + \beta_{ki}\delta_{ji}) = 0 \qquad (j,k=1,\cdots,m), \tag{11}$$

$$\tfrac{1}{2}a_{j0}\beta_{k0} + \sum_{i=1}^{m}(a_{ji}\delta_{ki} + \beta_{ki}\gamma_{ji}) = \begin{cases} 1 & (k=j) \\ 0 & (k\neq j) \end{cases} \tag{12}$$
$$(j=0,1,\cdots,m\,;\,k=1,\cdots,m).$$

S and S^{-1} have equal determinants, so that $|S| = \pm 1$. In fact, we take the variables in the order $\xi_0, \xi_1, \eta_1, \cdots, \xi_m, \eta_m$, and reflect on its main diagonal the determinant of S^{-1}, then interchange the second and third rows, the fourth and fifth rows, $\cdots$, and likewise the corresponding columns. The resulting determinant, aside from the factors 2 and $\frac{1}{2}$ which may be dropped, is identical with the determinant of S.

If in the above formulæ we drop the variable ξ_0 and the coefficients $a_{i0}, a_{0i}, \gamma_{i0}, \cdots$, we obtain results valid for the group $Q(m, F)$ of all linear transformations in F which leave invariant *

$$q_m \equiv \xi_1\eta_1 + \xi_2\eta_2 + \cdots + \xi_m\eta_m.$$

With such modifications, the following investigation of the structure of the group $G(m, F)$ will hold for the group $Q(m, F)$.

Let S be an arbitrary transformation of $G(m, F)$. We proceed to determine a transformation Σ, derived from

$$W_{i,j,\lambda},\ V_{i,j,\lambda},\ Q_{i,j,\lambda},\ X_{0,j,\lambda},\ Y_{0,j,\lambda},\ T_{i,\lambda},\ S_{ij},\ P_{ij}, \tag{13}$$

* For the case in which F is the Galois field of order p^n, this group was studied by the writer in the Proceedings of the London Mathematical Society, vol. 30 (1898), pp. 70–98. The calculations of pp. 77–80 are here avoided by the use of the simplicity of the group G_6.

which replaces ξ_1 by the same function that S does, viz.,

$$f_1 \equiv a_{10}\xi_0 + \sum_{j=1}^{m}(a_{1j}\xi_j + \gamma_{1j}\eta_j),$$

where by (10), for $j = k = 1$,

$$\tfrac{1}{4}a_{10}^2 + \sum_{i=1}^{m} a_{1i}\gamma_{1i} = 0. \tag{14}$$

The a_{1i}, $\gamma_{1i}(i = 1, \cdots, m)$ are not all zero, since otherwise $a_{10} = 0$ and $f_1 \equiv 0$.

(a) If $a_{11} \neq 0$, we may take for Σ the product

$$T_{1,\,a_{11}} X_{0,\,1,\,\frac{1}{2}a_{10}} Q_{1,\,2,\,a_{12}} W_{1,\,2,\,\gamma_{12}} \cdots Q_{1,\,m,\,a_{1m}} W_{1,\,m,\,\gamma_{1m}},$$

which replaces ξ_1 by

$$a_{10}\xi_0 + a_{11}\xi_1 - a_{11}^{-1}(\tfrac{1}{4}a_{10}^2 + a_{12}\gamma_{12} + \cdots + a_{1m}\gamma_{1m})\eta_1 + \sum_{j=2}^{m}(a_{1j}\xi_j + \gamma_{1j}\eta_j) \equiv f_1.$$

(b) If $\gamma_{11} \neq 0$, we may choose for Σ the product

$$S_{12} T_{1,\,\gamma_{11}} X_{0,\,1,\,\frac{1}{2}a_{10}} Q_{1,\,2,\,\gamma_{12}} W_{1,\,2,\,a_{12}} Q_{1,\,3,\,a_{13}} W_{1,\,3,\,\gamma_{13}} \cdots Q_{1,\,m,\,a_{1m}} W_{1,\,m,\,\gamma_{1m}}.$$

(c) Let $a_{1j} = \gamma_{1j} = 0\,(j = 1, \cdots, s-1)$, while a_{1s}, γ_{1s} are not both zero. By case (a) or (b), we obtain a transformation Σ' which replaces ξ_s by f_1. Then will $\Sigma \equiv \Sigma' P_{1s}$ replace ξ_1 by f_1.

We may therefore set $S = S\Sigma'$, where S' is a transformation of $G(m, F)$ which leaves ξ_1 fixed. Let S' replace η_1 by

$$f_1' \equiv \beta_{10}\xi_0 + \sum_{j=1}^{m}(\beta_{1j}\xi_j + \delta_{1j}\eta_j).$$

For S' we have $a_{10} = 0$, $a_{11} = 1$, $a_{1j} = 0\,(j = 2, \cdots, m)$, $\gamma_{1j} = 0\,(j = 1, \cdots, m)$. Then by (12) for $j = k = 1$, we have $\delta_{11} = 1$. By (11) for $j = k = 1$, we have

$$\tfrac{1}{4}\beta_{10}^2 + \sum_{i=1}^{m}\beta_{1i}\delta_{1i} = 0. \tag{15}$$

The transformation

$$\Sigma_1 \equiv Y_{0,\,1,\,\frac{1}{2}\beta_{10}} V_{2,\,1,\,-\beta_{12}} Q_{2,\,1,\,-\delta_{12}} \cdots V_{m,\,1,\,-\beta_{1m}} Q_{m,\,1,\,-\delta_{1m}}$$

leaves ξ_1 fixed and replaces η_1 by f_1'. We may therefore set

$$S' = \Sigma_1 S_1, \quad S \equiv \Sigma\,\Sigma_1 S_1,$$

where S_1 is a transformation of $G(m, F)$ which leaves ξ_1 and η_1 fixed. Hence S_1 is of the form S, written above, with

$$a_{10} = \beta_{10} = 0, \quad a_{11} = \delta_{11} = 1, \quad a_{1j} = \delta_{1j} = 0 \qquad (j = 2, \cdots, m),$$

$$\gamma_{1j} = \beta_{1j} = 0 \qquad (j = 1, \cdots, m).$$

Hence, by (11) and (12), for $j=1$, we get

$$\beta_{k1}=0, \quad \delta_{k1}=0 \qquad (k=2, \cdots, m).$$

By (10) and (12), for $k=1$, we get

$$\gamma_{j1}=0, \quad \alpha_{j1}=0 \qquad (j=0, 2, \cdots, m).$$

Hence S_1 is a transformation of $G(m, F)$ involving only the variables

$$\xi_0, \quad \xi_i, \quad \eta_i \qquad (i=2, \cdots, m).$$

We proceed with S_1 as we did with S. After $m-1$ such steps, we reach a transformation S_{m-1} affecting only ξ_0, ξ_m, η_m. Let it replace ξ_m by

$$f_m \equiv a_{m0}\xi_0 + a_{mm}\xi_m + \gamma_{mm}\eta_m,$$

where, by (10),

$$\tfrac{1}{4}a_{m0}^2 + a_{mm}\gamma_{mm}=0.$$

If $a_{mm} \neq 0$, the transformation $T_{m,\, a_{mm}} X_{0,\, m,\, \frac{1}{2}a_{m0}}$ replaces ξ_m by f_m. If $a_{mm}=0$, then $a_{m0}=0$ and $\gamma_{mm} \neq 0$, so that we may set

$$S_{m-1} \equiv (\xi_m \eta_m) T_{m,\, \gamma_{mm}} K,$$

where K leaves also ξ_m fixed.* Let K replace η_m by

$$f'_m \equiv \beta_{m0}\xi_0 + \beta_{mm}\xi_m + \delta_{mm}\eta_m.$$

By (11) and (12) for $i=j=k=m$, we get

$$\delta_{mm}=1, \quad \tfrac{1}{4}\beta_{m0}^2 + \beta_{mm}\delta_{mm}=0.$$

Hence $K = Y_{0,\, m,\, \frac{1}{2}\beta_{m0}} K'$, where K' leaves ξ_m and η_m fixed, and is therefore the identity or C_0, where C_0 alters only ξ_0 whose sign it changes. But

$$C_0 \equiv X_{0i1} Y_{0i1} X_{0i1} T_{i,\, -1}(\xi_i \eta_i). \tag{16}$$

It follows that an arbitrary transformation of $G(m, F)$ or of $Q(m, F)$ may be given one of the two forms A or $A(\xi_m \eta_m)$, where A is derived from the transformations (13) of determinant unity. Hence these groups contain subgroups of index 2, designated by $G'(m, F)$ and $Q'(m, F)$, generated by the transformations (13).

Consider, for $m \geqq 3$, the following subgroups of $G'(m, F)$ and $Q'(m, F)$:

$$G_1(m, F) \equiv \{S_{ij}, \quad W_{i,j,\lambda}, \quad X_{0,j,\lambda} \ (i, j=1, \cdots, m; i \neq j)\},$$

$$Q_1(m, F) \equiv \{S_{ij}, \quad W_{i,j,\lambda} \qquad (i, j=1, \cdots, m; i \neq j)\},$$†

* For the group $Q(m, F)$, K is necessarily the identity.

† For $m=3$, $Q_1(m, F)$ is the group G_6 of § 6.

where λ is arbitrary in the field F. By the formulæ of § 6, they contain the transformations

$$V_{i,j,\lambda}, \quad Q_{i,j,\lambda}, \quad P_{ij}T_{j,-1} \qquad (i,j=1,\cdots,m\,;\ i\neq j),$$

while $G_1(m, F)$ contains also $Y_{0,j,\lambda}$. Hence they contain

$$T_{1,\mu}T_{2,u} \equiv S_{12}P_{12}T_{1,-1}V_{1,2,\mu-1}W_{1,2,\mu}V_{1,2,\mu-1}, \tag{17}$$

$$T_{1,\mu^2} \equiv T_{1,\mu}T_{2,\mu}\cdot T_{2,\mu-1}T_{3,\mu-1}\cdot T_{3,\mu}T_{1,\mu}. \tag{18}$$

But $T_{k,\lambda}$ transforms $W_{i,j,\lambda}$, $V_{i,j,\lambda}$, $Q_{i,j,\lambda}$, $X_{0,j,\lambda}$, $Y_{0,j,\lambda}$, $T_{i,\lambda}$ into transformations of the same respective forms. Also

$$T_{i,\lambda}S_{ij} = S_{ij}T_{i,\lambda-1}, \quad T_{i,\lambda}P_{ij} = P_{ij}T_{j,\lambda}. \tag{19}$$

Hence every transformation of the group $G'(m, F)$ or $Q'(m, F)$ may be given one of the forms Σ, $\Sigma T_{m,\nu}$, where Σ belongs to $G_1(m, F)$ or $Q_1(m, F)$, respectively, while ν runs through the series of elements of F which are not-squares and whose ratios are all not-squares.

These results hold true for the group $G_1(2, F) \equiv G_5$ of § 6, viz.,

$$G_1(2, F) \equiv \{P_{12}T_{2,-1},\ S_{12},\ W_{1,2,\lambda},\ Q_{1,2,\lambda},\ X_{0,1,\lambda}\}.$$

Indeed, the latter group contains $Y_{0,1,\lambda}$, $X_{0,2,\lambda}$, $Y_{0,2,\lambda}$, $V_{1,2,\lambda}$, $Q_{2,1,\lambda}$, $T_{1,\mu}T_{2,\mu}$ and, by (16), $C_0(\xi_1\eta_1)T_{1,-1}$. The latter transforms $T_{1,\mu}T_{2,\mu}$ into $T_{1,\mu-1}T_{2,\mu}$, so that the product T_{2,μ^2} belongs to the group.

The group $G'(m, F)$, for $m \geqq 3$, contains the invariant subgroup $G_1(m, F)$; the group $Q'(m, F)$, for $m \geqq 2$, contains the invariant subgroup $Q_1(m, F)$. The invariant subgroup is extended to the main group by the right-hand extenders $T_{m,\nu}$, where ν runs through the series of those not-squares of F, the ratio of no two of which is a square.

§ 8. *Structure of the groups $G_1(m, F)$ and $Q_1(m, F)$.*

By § 6 we have the results:*

The group $G_1(2, F)$ on five variables is simple. The senary group $Q_1(3, F)$ is simple or has the maximal invariant subgroup composed of the identity I and the transformation $T_{1,-1}T_{2,-1}T_{3,-1}$ according as -1 is a not-square or a square in the field F.

We employ these theorems in dealing with the case of general m. Let J be an invariant subgroup of $G_1(m, F)$ containing a transformation S not the identity I. To treat simultaneously the group $Q_1(m, F)$, let J be an invariant subgroup containing a transformation S neither the identity nor

$$T \equiv T_{1,-1}T_{2,-1}\cdots T_{m,-1},$$

* If there be a modulus p, we assume, in this section, that $p \neq 2$.

in case the latter belongs to $Q_1(m, F)$. The groups $G_1(m, F)$ and $Q_1(m, F)$ will be considered together under the notation G. We assume that $m \geqq 3$.

LEMMA I. *The group J contains a transformation which multiplies ξ_1 by a constant and does not reduce to I or T.*

By hypothesis, J contains a transformation S neither I nor T. Let S replace ξ_1 by

$$f_1 \equiv a_{10}\xi_0 + \sum_{j=1}^{m}(a_{1j}\xi_j + \gamma_{1j}\eta_j) \neq a_{11}\xi_1,$$

the coefficients being subject to the condition (14).

(*a*) If $\gamma_{11} \neq 0$, the group G contains the product

$$P \equiv T_{1,\,\gamma_{11}^{-1}} T_{2,\,\gamma_{11}^{-1}} Y_{0,\,1,\,\frac{1}{2}a_{10}} V_{2,\,1,\,-\gamma_{11}a_{12}} Q_{2,\,1,\,-\gamma_{11}^{-1}\gamma_{12}} K,$$

where K denotes the transformation

$$K \equiv V_{3,\,1,\,-a_{13}} Q_{3,\,1,\,-\gamma_{13}} \cdots V_{m,\,1,\,-a_{1m}} Q_{m,\,1,\,-\gamma_{1m}}.$$

Employing (14), we find that P replaces ξ_1 by $\gamma_{11}^{-1}\xi_1$ and η_1 by f_1. Hence J contains $S_1 \equiv P^{-1}SP$, which replaces ξ_1 by $\gamma_{11}^{-1}\eta_1$.

If S_1 multiplies ξ_2 by a constant, J contains its transform S_1' by $P_{12}T_{2,\,-1}$. This S_1' multiplies ξ_1 by a constant and is neither I nor T.

If S_1 does not multiply ξ_2 by a constant, there exists in G a transformation B leaving ξ_1 and η_1 unaltered and not commutative with S_1, so that J contains $S_1^{-1}B^{-1}S_1B$, which leaves ξ_1 fixed and is neither I nor T. In fact, if S_1 be commutative with $V_{2,\,3,\,\lambda}$, we find, on equating the expressions by which $S_1V_{2,\,3,\,\lambda}$ and $V_{2,\,3,\,\lambda}S_1$ replace η_3, that

$$\eta_3' - \lambda\xi_2' = \eta_3' + (\ )\xi_2 + (\ )\xi_3.$$

Similarly, if S_1 be commutative with $Q_{3,\,2,\,\lambda}$, we get

$$\xi_3' + \lambda\xi_2' = \xi_3' + (\ )\xi_2 + (\ )\eta_3.$$

Hence would $\xi_2' = (\ )\xi_2$, contrary to hypothesis.

(*b*) If $\gamma_{11} = 0$, we may take* $a_{12} \neq 0$. Then G contains

$$R \equiv T_{2,\,a_{12}} T_{3,\,a_{12}} X_{0,\,2,\,\frac{1}{2}a_{10}} Q_{2,\,1,\,a_{11}} Q_{2,\,3,\,a_{12}^{-1}a_{13}} W_{2,\,3,\,a_{12}\gamma_{13}} K_1,$$

$$K_1 \equiv Q_{2,\,4,\,a_{14}} W_{2,\,4,\,\gamma_{14}} \cdots Q_{2,\,m,\,a_{1m}} W_{2,\,m,\,\gamma_{1m}}.$$

Employing (14), we find that R replaces ξ_2 by f_1 without altering ξ_1. Then J contains $S_2 \equiv R^{-1}SR$ which replaces ξ_1 by ξ_2.

If S_2 multiplies ξ_3 by a constant, J contains its transform S_2' by $P_{13}T_{1,\,-1}$. But S_2' multiplies ξ_1 by a constant and is neither I nor T.

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*If $a_{12} = a_{13} = \cdots = a_{1m} = 0$, then $a_{10} = 0$ by (14). Not every γ_{1j} is zero by hypothesis. If $\gamma_{12} \neq 0$, for example, we take in place of S its transform by S_{23}, for which $a_{12}' \neq 0$.

If S_2 does not multiply ξ_3 by a constant, S_2 is not commutative with both $V_{3,1,\lambda}$ and $V_{3,2,\lambda}$, since $S_2 V_{3,k,\lambda}$ and $V_{3,k,\lambda} S_2$ replace η_k by $\eta_k' - \lambda\xi_3'$ and $\eta_k' + (\ )\xi_3 + (\ )\xi_k$, respectively. Hence J contains

$$S_2^{-1} V_{3,k,\lambda}^{-1} S_2 V_{3,k,\lambda} \qquad (k=1, 2)$$

which leave ξ_1 fixed (so that neither is T), and are not both the identity.

LEMMA II. *The group J contains a transformation which leaves ξ_1 and η_1 unaltered and is different from the identity.*

In view of Lemma I, we may suppose that J contains a transformation S, different from I and T, which replaces ξ_1 by $a\xi_1$ and η_1 by

$$\beta_{10}\xi_0 + \sum_{j=1}^{m} (\beta_{1j}\xi_j + \delta_{1j}\eta_j),$$

where by (12), for $j = k = 1$, we have $\delta_{11} = a^{-1}$, and by (11), for $j = k = 1$,

$$\tfrac{1}{4}\beta_{10}^2 + \sum_{i=1}^{m} \beta_{1i}\delta_{1i} = 0. \tag{15}$$

(*A*) Let $\beta_{11} = 0$, $\beta_{1j} = \delta_{1j} = 0 (j = 2, \cdots, m)$. Then $\beta_{10} = 0$ by (15). Hence $S = T_{1,a} S_1$, where S_1 leaves ξ_1 and η_1 unaltered. Hence,* S_1 involves only the variables ξ_0, ξ_i, $\eta_i (i = 2, \cdots, m)$. If $a = 1$, the Lemma is proved. Let next $a \neq 1$.

If $S_1 = I$, or if, when $G \equiv Q_1(m, F)$, $S = T_{2,-1} T_{3,-1} \cdots T_{m,-1} \equiv \tau$, the group J contains $T_{1,a}$ or $S^2 = T_{1,a^2}$ respectively. In the second case, $a \neq -1$, since $S \neq T$. Transforming by $P_{12} T_{2,-1}$, we obtain in either case a transformation leaving ξ_1 and η_1 fixed and not the identity.

If S_1 be neither I nor τ, there exists in G a transformation Σ_1 affecting the same variables as S_1 and not commutative with S_1.† Hence J contains

$$S^{-1}\Sigma_1^{-1} S \Sigma_1 \equiv S_1^{-1}\Sigma_1^{-1} S_1 \Sigma_1 \neq I,$$

which leaves ξ_1 and η_1 unaltered.

(*B*) Let $\beta_{11} = 0$, and β_{1j}, $\delta_{1j} (j = 2, \cdots, m)$ be not all zero. Then, by § 7, G contains a transformation L which leaves ξ_1 and η_1 fixed and replaces ξ_2 by

$$\beta_{10}\xi_0 + \sum_{j=2}^{m} (\beta_{1j}\xi_j + \delta_{1j}\eta_j) \qquad \left[\tfrac{1}{4}\beta_{10}^2 + \sum_{j=2}^{m} \beta_{1j}\delta_{1j} = 0\right].$$

Hence J contains $S_1 = L^{-1} S L$, which replaces ξ_1 by $a\xi_1$ and η_1 by $\xi_2 + a^{-1}\eta_1$. The latter function is invariant under the transformations $Q_{3,2,\lambda}$, $V_{2,3,\lambda}$, T_{3,μ^2} and $T_{1,\lambda} T_{2,\lambda-1}$ belonging to G. If any one of these, say Σ, is not commuta-

* See § 7, case (*c*).

† For $G \equiv G_1(m, F)$, the transformation changing the signs of $\xi_0, \xi_2, \eta_2, \cdots, \xi_m, \eta_m$ is of determinant -1 and hence does not belong to G.

tive with S_1, then J contains $S_1^{-1}\Sigma^{-1}S_1\Sigma \neq I$, which leaves ξ_1 and η_1 fixed. Suppose next that they are all commutative with S_1. Equating the two expressions by which $S_1 V_{2,3,\lambda}$ and $V_{2,3,\lambda}S_1$ replace η_2, and the two by which they replace η_3, we get

$$\xi_3' = \delta_{22}\xi_3 - \delta_{23}\xi_2, \quad \xi_2' = -\delta_{32}\xi_3 + \delta_{33}\xi_2.$$

Equating the expressions by which $S_1 Q_{3,2,\lambda}$ and $Q_{3,2,\lambda}S_1$ replace ξ_3, and those by which they replace η_2, we find that

$$\xi_2' = a_{33}\xi_2 - \gamma_{32}\eta_3, \quad \eta_3' = -\beta_{23}\xi_2 + \delta_{22}\eta_3.$$

The field F contains an element λ different from 0 and 1. If we equate the expressions by which $T_{1,\lambda}T_{2,\lambda-1}S_1$ and $S_1 T_{1,\lambda}T_{2,\lambda-1}$ replace η_2, we find that $\beta_{23} = \delta_{23} = 0$. Hence S_1 merely multiplies ξ_3 and η_3 by the same constant δ_{22}. Transforming S_1 by $P_{13}T_{1,-1}$, we obtain a transformation of the kind treated in case (A).

(C) Let $\beta_{11} \neq 0$, β_{1j}, δ_{1j} $(j = 2, \cdots, m)$ be not all zero. By a simple transformation, we may take $\delta_{12} \neq 0$. Transforming S by $T_{2,\delta_{12}}T_{3,\delta_{12}}$, we reach a transformation S' with $\beta_{11} \neq 0$, $\delta_{12} = 1$. Then

$$\omega \equiv Y_{0,2,-\frac{1}{2}\beta_{10}}Q_{3,2,-\delta_{13}}V_{2,3,\beta_{13}}\cdots Q_{m,2,-\delta_{1m}}V_{2,m,\beta_{1m}}$$

leaves ξ_1, η_1, and ξ_2 unaltered and replaces η_2 by

$$\beta_{10}\xi_0 + (\beta_{11}\delta_{11} + \beta_{12})\xi_2 + \eta_2 + \sum_{j=3}^{m}(\beta_{1j}\xi_j + \delta_{1j}\eta_j).$$

Then J contains $S_1 \equiv \omega^{-1}S'\omega$, which replaces ξ_1 by $a\xi_1$, and η_1 by

$$\beta_{11}\xi_1 + a^{-1}\eta_1 - a^{-1}\beta_{11}\xi_2 + \eta_2.$$

Let $\mu \equiv -\beta_{11}a^{-1} \neq 0$. If among the transformations $Q_{3,2,\mu}W_{2,3,1} \equiv \sigma$, $T_{3,\mu}T_{2,\mu}S_{23}$, etc., of G, which leave ξ_1, η_1, and $\mu\xi_2 + \eta_2$ invariant, there exists one, say R, which is not commutative with S_1, then J contains $S_1^{-1}R^{-1}S_1R$ which leaves ξ_1 and η_1 fixed and differs from the identity. In the contrary case, we find, on equating the functions by which $S_1\sigma$ and σS_1 replace ξ_2, that

$$\eta_3' = (\ )\eta_3 - a_{23}\eta_2 + \mu a_{23}\xi_2.$$

Then, by (11) for $j = k = 3$, we have $a_{23} = 0$. Finally, if S_1 be commutative with $T_{3,\mu}T_{2,\mu}S_{23}$, it must multiply ξ_3 and η_3 by the same constant.

(D) Let $\beta_{11} \neq 0$, $\beta_{1j} = \delta_{1j} = 0\,(j = 2, \cdots, m)$. Then, by (15),

$$\tfrac{1}{4}\beta_{10}^2 + \beta_{11}\delta_{11} = 0, \quad \delta_{11} = a^{-1}.$$

Hence S replaces ξ_1 by $a\xi_1$ and replaces η_1 by

$$\beta_{10}\xi_0 + \beta_{11}\xi_1 + a^{-1}\eta_1 = a^{-1}(\eta_1 + 2\lambda\xi_0 - \lambda^2\xi_1),$$

if we set $2\lambda \equiv a\beta_{10}$. Hence $S = Y_{0,1,\lambda} T_{1,a} S_1$, where S_1 leaves ξ_1 and η_1 unaltered and so involves only $\xi_0, \xi_i, \eta_i (i = 2, \cdots, m)$. Then

$$T_{1,\rho}^{-1} S T_{1,\rho} \cdot S^{-1} \equiv Y_{0,1,\lambda\rho^{-1}-\lambda}$$

belongs to J and is not the identity if $\rho \neq 1$. Transforming it by $P_{12} T_{2,-1}$, we obtain in J a transformation $\neq I$, which leaves ξ_1 and η_1 unaltered.

In the proofs of lemmas I and II, we assumed the existence of the variables $\xi_0, \xi_1, \eta_1, \xi_2, \eta_2, \xi_3, \eta_3$ only. If $m \geqq 4$, we may therefore conclude that J contains a transformation different from I and T which leaves $\xi_1, \eta_1, \xi_2, \eta_2$ unaltered. After $m - 2$ applications of the lemmas, we reach in J a transformation, neither I nor T, which affects only $\xi_0, \xi_{m-1}, \eta_{m-1}, \xi_m, \eta_m$. Transforming it by $P_{1\,m-1} P_{2m}$, which belongs to G, we obtain a transformation, neither I nor T, which affects only $\xi_0, \xi_1, \eta_1, \xi_2, \eta_2$. From the simplicity of $G_1(2, F)$, it follows, when $G \equiv G_1(m, F)$, that J contains all the transformations of $G_1(2, F)$. Transforming them by the $P_{ij} T_{j,-1}$, we reach in J all the generators of $G_1(m, F)$. Since $Q_1(3, F)$ is simple or has the maximal invariant subgroup $\{I, T\}$, it follows, when $G \equiv Q_1(m, F)$, that J contains the generators of $Q_1(3, F)$ and, therefore, by transformation by the $P_{ij} T_{j,-1}$, the generators of $Q_1(m, F)$.

If $m \geqq 2$, the group $G_1(m, F)$ is simple. If $m \geqq 3$, the group $Q_1(m, F)$ is simple or has the maximal invariant subgroup $\{I, T\}$, according as -1 is a not-square or a square in the field F.

§9. *Definition and generators of a subgroup of* $G_1(3, F)$.

We next define and investigate the septenary group in an arbitrary* field F which becomes, for the case of a continuous field, the continuous group of fourteen parameters studied by KILLING, ENGEL,† and CARTAN.‡

The totality of linear homogeneous transformations S on seven variables with coefficients in F which leave absolutely invariant

$$g_3 \equiv \xi_0^2 + \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3$$

form a group $G(3, F)$. Taking $m = 3$, and giving S the notation of §7, we may take as the conditions upon S the relations (9), (10), (11), (12). We study the group H of transformations S, belonging to $G(3, F)$, which, when operating cogrediently upon the two sets of variables

$$(20) \qquad \xi_0, \xi_1, \eta_1, \xi_2, \eta_2, \xi_3, \eta_3; \quad \bar{\xi}_0, \bar{\xi}_1, \bar{\eta}_1, \bar{\xi}_2, \bar{\eta}_2, \bar{\xi}_3, \bar{\eta}_3,$$

* As in §§7–8, we assume that $p \neq 2$, if F has a modulus p.

† LIE-ENGEL, *Transformationsgruppen*, vol. 3, pp. 763–765, 777.

‡ CARTAN, *Thèses*, Paris, 1894, pp. 146, 149–151.

leave invariant the system of equations

$$(21)\qquad \begin{cases} X_1 + Y_{23} = 0, & X_2 + Y_{31} = 0, & X_3 + Y_{12} = 0, \\ Y_1 + X_{23} = 0, & Y_2 + X_{31} = 0, & Y_3 + X_{12} = 0. \end{cases}$$

We have here employed the notations

$$X_i \equiv \begin{vmatrix} \xi_0 & \xi_i \\ \bar{\xi}_0 & \bar{\xi}_i \end{vmatrix}, \quad Y_i \equiv \begin{vmatrix} \xi_0 & \eta_i \\ \bar{\xi}_0 & \bar{\eta}_i \end{vmatrix}, \quad X_{ij} \equiv \begin{vmatrix} \xi_i & \xi_j \\ \bar{\xi}_i & \bar{\xi}_j \end{vmatrix}, \quad Y_{ij} \equiv \begin{vmatrix} \eta_i & \eta_j \\ \bar{\eta}_i & \bar{\eta}_j \end{vmatrix}, \quad Z_{ij} \equiv \begin{vmatrix} \xi_i & \eta_j \\ \bar{\xi}_i & \bar{\eta}_j \end{vmatrix}.$$

Multiplying the equations in the first row of (21) by η_1, η_2, η_3, respectively, and adding the results, we get $X_1\eta_1 + X_2\eta_2 + X_3\eta_3 = 0$, since

$$Y_{23}\eta_1 + Y_{31}\eta_2 + Y_{12}\eta_3 \equiv \begin{vmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_1 & \eta_2 & \eta_3 \\ \bar{\eta}_1 & \bar{\eta}_2 & \bar{\eta}_3 \end{vmatrix} \equiv 0.$$

Similarly from the equations in the second row of (21), we get

$$Y_1\xi_1 + Y_2\xi_2 + Y_3\xi_3 = 0.$$

In view of the identity

$$Y_i\xi_i - X_i\eta_i \equiv \xi_0(\xi_i\bar{\eta}_i - \xi_i\eta_i) \equiv \xi_0 Z_{ii},$$

we derive the equation* [a consequence of equations (21)]

$$(22)\qquad Z_{11} + Z_{22} + Z_{33} = 0.$$

An inspection of equations (21) and (22) leads to a proof of the following

LEMMA: *A linear equation involving X_i, Y_i, X_{ij}, Y_{ij}, Z_{ij} will be a consequence of equations* (21) *if, and only if, the coefficient of X_i equals that of Y_{jk}, the coefficient of Y_i equals that of X_{jk}, the coefficients of Z_{11}, Z_{22}, Z_{33} are all equal, and the coefficient of each $Z_{r,s}$ $(r, s = 1, 2, 3;\ r \neq s)$ is zero; where i, j, k is any cyclic permutation of $1, 2, 3$.*

When operating cogrediently upon the variables (20), the transformation

$$S: \begin{cases} \xi_i' = a_{i0}\xi_0 + a_{i1}\xi_1 + a_{i2}\xi_2 + a_{i3}\xi_3 + \gamma_{i1}\eta_1 + \gamma_{i2}\eta_2 + \gamma_{i3}\eta_3 & (i = 0, 1, 2, 3) \\ \eta_i' = \beta_{i0}\xi_0 + \beta_{i1}\xi_1 + \beta_{i2}\xi_2 + \beta_{i3}\xi_3 + \delta_{i1}\eta_1 + \delta_{i2}\eta_2 + \delta_{i3}\eta_3 & (i = 1, 2, 3) \end{cases}$$

* Other equations may be derived from (21); for example,

$$Y_{12}X_{12} + Y_{23}X_{23} + Y_{31}X_{31} = 0, \quad Y_{12}^2 + Z_{22}X_{22} + Z_{31}X_{31} = 0.$$

Since they are quadratic in each set of variables $\xi, \eta; \bar{\xi}, \bar{\eta}$, they do not enter into the discussion of the invariance of equations (21).

replaces the function X_l by the expression

$$\sum_{i=1}^{3} \begin{vmatrix} a_{00} & a_{0i} \\ a_{l0} & a_{li} \end{vmatrix} X_i + \sum_{i=1}^{3} \begin{vmatrix} a_{00} & \gamma_{0i} \\ a_{l0} & \gamma_{li} \end{vmatrix} Y_i + \sum_{i,j}^{1,2,3} \begin{vmatrix} a_{0i} & \gamma_{0j} \\ a_{li} & \gamma_{lj} \end{vmatrix} Z_{ij}$$
$$+ \sum_{\substack{i,j=1,2,3 \\ i<j}} \left\{ \begin{vmatrix} a_{0i} & a_{0j} \\ a_{li} & a_{lj} \end{vmatrix} X_{ij} + \begin{vmatrix} \gamma_{0i} & \gamma_{0j} \\ \gamma_{li} & \gamma_{lj} \end{vmatrix} Y_{ij} \right\}.$$

Let l, m, n be any cyclic permutation of $1, 2, 3$. Then S replaces Y_{mn} by

$$\sum_{i=1}^{3} \begin{vmatrix} \beta_{m0} & \beta_{mi} \\ \beta_{n0} & \beta_{ni} \end{vmatrix} X_i + \sum_{i=1}^{3} \begin{vmatrix} \beta_{m0} & \delta_{mi} \\ \beta_{n0} & \delta_{ni} \end{vmatrix} Y_i + \sum_{i,j} \begin{vmatrix} \beta_{mi} & \delta_{mj} \\ \beta_{ni} & \delta_{nj} \end{vmatrix} Z_{ij}$$
$$+ \sum_{i,j} \left\{ \begin{vmatrix} \beta_{mi} & \beta_{mj} \\ \beta_{ni} & \beta_{nj} \end{vmatrix} X_{ij} + \begin{vmatrix} \delta_{mi} & \delta_{mj} \\ \delta_{ni} & \delta_{nj} \end{vmatrix} Y_{ij} \right\}.$$

Hence S replaces the equation $X_l + Y_{mn} = 0$ by an equation which is linear in $X_i, Y_i, X_{ij}, Y_{ij}, Z_{ij}$. Applying the above lemma, we see that the resulting equation will be a consequence of equations (21) if, and only if, the conditions (23), (24), (25) below are satisfied. Similarly, S replaces the equation $Y_l + X_{mn} = 0$ by an equation which is linear in $X_i, Y_i, X_{ij}, Y_{ij}, Z_{ij}$ with coefficients obtained from the corresponding coefficients in the earlier equation by interchanging a_{r0} with β_{r0}, a_{ri} with β_{ri}, γ_{ri} with δ_{ri}, for $r = 1, 2, 3$; $i = 1, 2, 3$. In view of the lemma, the resulting equation will be a consequence of equations (21), if, and only if, the conditions (26), (27), (28) are satisfied.

(23) $$\begin{vmatrix} a_{00} & a_{0i} \\ a_{l0} & a_{li} \end{vmatrix} + \begin{vmatrix} \beta_{m0} & \beta_{mi} \\ \beta_{n0} & \beta_{ni} \end{vmatrix} = \begin{vmatrix} \gamma_{0j} & \gamma_{0k} \\ \gamma_{lj} & \gamma_{lk} \end{vmatrix} + \begin{vmatrix} \delta_{mj} & \delta_{mk} \\ \delta_{nj} & \delta_{nk} \end{vmatrix}.$$

(24) $$\begin{vmatrix} a_{00} & \gamma_{0i} \\ a_{l0} & \gamma_{li} \end{vmatrix} + \begin{vmatrix} \beta_{m0} & \delta_{mi} \\ \beta_{n0} & \delta_{ni} \end{vmatrix} = \begin{vmatrix} a_{0j} & a_{0k} \\ a_{lj} & a_{lk} \end{vmatrix} + \begin{vmatrix} \beta_{mj} & \beta_{mk} \\ \beta_{nj} & \beta_{nk} \end{vmatrix}.$$

(25) $$\begin{vmatrix} a_{0i} & \gamma_{0j} \\ a_{li} & \gamma_{lj} \end{vmatrix} + \begin{vmatrix} \beta_{mi} & \delta_{mj} \\ \beta_{ni} & \delta_{nj} \end{vmatrix} \equiv C_{ij}, \quad \begin{cases} C_{11} = C_{22} = C_{33}. \\ C_{ij} = 0 \end{cases} \quad (i, j = 1, 2, 3; i \neq j).$$

(26) $$\begin{vmatrix} a_{00} & a_{0i} \\ \beta_{l0} & \beta_{li} \end{vmatrix} + \begin{vmatrix} a_{m0} & a_{mi} \\ a_{n0} & a_{ni} \end{vmatrix} = \begin{vmatrix} \gamma_{0j} & \gamma_{0k} \\ \delta_{lj} & \delta_{lk} \end{vmatrix} + \begin{vmatrix} \gamma_{mj} & \gamma_{mk} \\ \gamma_{nj} & \gamma_{nk} \end{vmatrix}.$$

(27) $$\begin{vmatrix} a_{00} & \gamma_{0i} \\ \beta_{l0} & \delta_{li} \end{vmatrix} + \begin{vmatrix} a_{m0} & \gamma_{mi} \\ a_{n0} & \gamma_{ni} \end{vmatrix} = \begin{vmatrix} a_{0j} & a_{0k} \\ \beta_{lj} & \beta_{lk} \end{vmatrix} + \begin{vmatrix} a_{mj} & a_{mk} \\ a_{nj} & a_{nk} \end{vmatrix}.$$

(28) $$\begin{vmatrix} a_{0i} & \gamma_{0j} \\ \beta_{li} & \delta_{lj} \end{vmatrix} + \begin{vmatrix} a_{mi} & \gamma_{mj} \\ a_{ni} & \gamma_{nj} \end{vmatrix} = d_{ij}, \quad \begin{cases} d_{11} = d_{22} = d_{33}, \\ d_{ij} = 0 \end{cases} \quad (i, j = 1, 2, 3; i \neq j).$$

The formulæ hold for any cyclic permutations i, j, k; l, m, n of $1, 2, 3$.

The transformations S of the group H may be defined by the conditions (9), (10), (11), (12) *and* (23)–(28).

Among the transformations of H occur the simple types: *

$$Q_{i,j,\tau},\quad T_{i,\tau^{-2}}T_{j,\tau}T_{k,\tau},\quad X_{0,i,\tau}V_{j,k,\tau},\quad Y_{0,i,\tau}W_{j,k,\tau}, \tag{29}$$

where τ is arbitrary in the field F, and i, j, k form any cyclic permutation of $1, 2, 3$. From them we obtain, as in § 6,

$$T_{i,-1}P_{ij} \equiv P_{ij}T_{j,-1} = Q_{j,i,-1}Q_{i,j,1}Q_{j,i,-1};$$

$$\begin{aligned} &X_{0,1,1}V_{2,3,1}\cdot Y_{0,1,1}W_{2,3,1}\cdot X_{0,1,1}V_{2,3,1} \\ &\quad = X_{0,1,1}Y_{0,1,1}X_{0,1,1}\cdot V_{2,3,1}W_{2,3,1}V_{2,3,1} \\ &\quad = C_0(\xi_1\eta_1)T_{1,-1}\cdot T_{2,-1}P_{23}S_{23} = (\xi_1\eta_1)(\xi_2\eta_2)(\xi_3\eta_3)C_0T_{1,-1}T_{2,-1}P_{23}, \end{aligned}$$

upon applying formulæ (16), (17), the latter with $\mu = 1$, and with $2, 3$ in place of $1, 2$. Since $T_{2,-1}P_{23}$ belongs to H, it follows that H contains

$$\Sigma \equiv (\xi_1\eta_1)(\xi_2\eta_2)(\xi_3\eta_3)C_0T_{1,-1}. \tag{30}$$

The transformations $T_{i,\tau}T_{j,\tau^{-1}}$ leave invariant g_3 and the equations (21), and therefore belong to the group H. From the transformations (29) we readily derive $T_{i,\tau}T_{j,\tau^{-1}}$, for the case in which τ is a cube in the field.

THEOREM: *The transformations of H of determinant $+1$ form a subgroup H' of index 2 which is generated by the transformations*

$$Q_{i,j,\tau},\quad T_{i,\tau}T_{j,\tau^{-1}},\quad X_{0,i,\tau}V_{j,k,\tau},\quad Y_{0,i,\tau}W_{j,k,\tau}. \tag{31}$$

Let S be an arbitrary transformation of H and let it be exhibited in the above form. We are to prove that there exists a transformation K derived from the types (31) such that $KS = I$, the identity, or $C_0T_{3,-1}$, the latter of determinant -1.

We may assume that $a_{11} \neq 0$ in S. For, if $a_{1i} \neq 0$, the product $T_{i,-1}P_{i1}S$ has $a_{11} \neq 0$; if $\gamma_{1i} \neq 0$, the product ΣS has $a_{1i} \neq 0$; while the case $a_{1i} = \gamma_{1i} = 0$ $(i = 1, 2, 3)$ is excluded, since then $a_{10} = 0$ by (14). The product $S_1 \equiv T_{1,a_{11}^{-1}}T_{2,a_{11}}S$ replaces ξ_1 by a function of the form

$$f \equiv a_{10}\xi_0 + \xi_1 + a_{12}\xi_2 + a_{13}\xi_3 + \gamma_{11}\eta_1 + \gamma_{12}\eta_2 + \gamma_{13}\eta_3.$$

Then $Y_{0,2,\gamma_{13}}W_{3,1,\gamma_{13}}S_1 \equiv S_2$ replaces ξ_1 by a function of the form f with $\gamma_{13} = 0$. Then $Q_{1,3,-a_{13}}S_2 \equiv S_3$ replaces ξ_1 by a function f with $\gamma_{13} = a_{13} = 0$. Next, $Q_{1,2,-a_{12}}S_3 \equiv S_4$ replaces ξ_1 by a function

$$f' \equiv a_{10}\xi_0 + \xi_1 + \gamma_{11}\eta_1 + \gamma_{12}\eta_2.$$

Then, by (14), we have $\frac{1}{4}a_{10}^2 + \gamma_{11} = 0$. If $\gamma_{11} \neq 0$, $Q_{2,1,\kappa}S_4 \equiv S_5$ replaces ξ_1

* They correspond respectively to the generators X_{ij}, X_{i0}, X_{0i}, $-X_{00}$ of CARTAN, p. 146.

by a function of the form f' with $\gamma_{12}-\kappa\gamma_{11}$ in place of γ_{12}. By a proper choice of κ, we make the new γ_{12} equal to zero. Hence S_5 replaces ξ_1 by

$$\xi_1+a_{10}\xi_0-\tfrac{1}{4}a_{10}^2\eta_1,$$

so that $(X_{0,1,\frac{1}{2}a_{10}}V_{2,3,\frac{1}{2}a_{10}})^{-1}S_5$ leaves ξ_1 unaltered. But, if $\gamma_{11}=0$, then $f'\equiv\xi_1+\gamma_{12}\eta_2$, so that $Y_{0,3,-\gamma_{12}}W_{1,2,-\gamma_{12}}S_4$ leaves ξ_1 unaltered.

Consider a transformation S' which replaces ξ_1 by ξ_1, and η_1 by

$$f_1\equiv\beta_{10}\xi_0+\beta_{11}\xi_1+\eta_1+\beta_{12}\xi_2+\beta_{13}\xi_3+\delta_{12}\eta_2+\delta_{13}\eta_3.$$

Then, by applying in succession as left-hand multipliers $Q_{3,1,\delta_{13}}$, $X_{0,2,\beta_{13}}V_{3,1,\beta_{13}}$, $Q_{2,1,\delta_{12}}$, we obtain a transformation S'' which replaces ξ_1 by ξ_1, and η_1 by $\beta_{10}\xi_0+\beta_{11}\xi_1+\eta_1+\beta_{12}\xi_2$. By (15), $\beta_{11}=-\frac{1}{4}\beta_{10}^2$. If $\beta_{11}=0$, S'' replaces η_1 by $\eta_1+\beta_{12}\xi_2$, so that

$$X_{0,3,-\beta_{12}}V_{1,2,-\beta_{12}}S''$$

leaves ξ_1 and η_1 unaltered. If $\beta_{11}\neq 0$,

$$(Y_{0,1,\frac{1}{2}\beta_{10}}W_{2,3,\frac{1}{2}\beta_{10}})^{-1}S''\equiv S'''$$

replaces ξ_1 by ξ_1, and η_1 by $\eta_1+\beta_{12}\xi_2-\frac{1}{2}\beta_{10}\eta_3$. Then $Q_{3,1,-\frac{1}{2}\beta_{10}}S'''$ replaces ξ_1 by ξ_1, and η_1 by $\eta_1+\beta_{12}\xi_2$, a case just considered.

It remains to discuss those substitutions S of H which do not alter ξ_1 and η_1 and therefore have (compare § 7)

$$a_{11}=\delta_{11}=1,\quad a_{12}=a_{13}=a_{21}=a_{31}=0,\quad \delta_{12}=\delta_{13}=\delta_{21}=\delta_{31}=0,$$

$$a_{01}=a_{10}=\beta_{10}=\gamma_{01}=0,\quad \beta_{1i}=\beta_{i1}=\gamma_{1i}=\gamma_{i1}=0\quad (i=1,2,3).$$

By (25), for $(l,i,j)=(1,1,2),(1,1,3),(2,2,1),(2,3,1),(3,2,1),(3,3,1)$, we get respectively

$$\gamma_{02}=0,\quad \gamma_{03}=0,\quad \beta_{32}=0,\quad \beta_{33}=0,\quad \beta_{22}=0,\quad \beta_{23}=0.$$

By (28), for $(l,i,j)=(1,2,1),(1,3,1),(2,1,2),(2,1,3),(3,1,2),(3,1,3)$, we get respectively

$$a_{02}=0,\quad a_{03}=0,\quad \gamma_{32}=0,\quad \gamma_{33}=0,\quad \gamma_{22}=0,\quad \gamma_{23}=0.$$

By (24), for $(l,i)=(3,1)$, $\beta_{20}=0$; for $(l,i)=(2,1)$, $\beta_{30}=0$. By (26), for $(l,i)=(3,1)$, $a_{20}=0$; for $(l,i)=(2,1)$, $a_{30}=0$. By (27), for $(l,i)=(3,2),(2,3),(3,3),(2,2)$, we get respectively

$$-a_{23}=a_{00}\delta_{32},\quad -a_{32}=a_{00}\delta_{23},\quad a_{22}=a_{00}\delta_{33},\quad a_{33}=a_{00}\delta_{22}.$$

Finally, by (23), for $(l,i)=(1,1)$, and by (9), we get respectively

$$(32)\qquad \begin{vmatrix}\delta_{22} & \delta_{23}\\ \delta_{32} & \delta_{33}\end{vmatrix}=a_{00},\quad a_{00}^2=1.$$

Hence S has the following form:

$$\xi_0' = a_{00}\xi_0, \quad \eta_2' = \delta_{22}\eta_2 + \delta_{23}\eta_3, \quad \xi_2' = a_{00}\delta_{33}\xi_2 - a_{00}\delta_{32}\xi_3,$$
$$\eta_3' = \delta_{32}\eta_2 + \delta_{33}\eta_3, \quad \xi_3' = -a_{00}\delta_{23}\xi_2 + a_{00}\delta_{22}\xi_3,$$

subject to relations (32). Its determinant equals

$$a_{00}^3 \begin{vmatrix} \delta_{22} & \delta_{23} \\ \delta_{32} & \delta_{33} \end{vmatrix}^2 = a_{00} = \pm 1.$$

Denote by S_1 the preceding substitution S for the case $a_{00} = +1$. For $a_{00} = -1$, $S = S' C_0 T_{3,-1}$, where S' is of the form S_1, of determinant $+1$.

The general substitution S_1 has the form

$$\eta_2' = \kappa\eta_2 + \lambda\eta_3, \quad \eta_3' = \mu\eta_2 + \nu\eta_3, \quad \xi_2' = \nu\xi_2 - \mu\xi_3, \quad \xi_3' = -\lambda\xi_2 + \kappa\xi_3, \tag{33}$$
$$\kappa\nu - \lambda\mu = 1.$$

If $\kappa = \nu = 0$, then $S_1 = T_{2,\mu-1}T_{3,\mu}T_{3,-1}P_{23}$. If κ, ν are not both zero, we may take $\nu \neq 0$; for, if $\nu = 0$, $\kappa \neq 0$, the transform of S_1 by $T_{3,-1}P_{23}$ has $\nu \neq 0$. Then $Q_{2,3,\mu\nu^{-1}}S_1$ may be expressed as the product

$$Q_{3,2,-\lambda\nu^{-1}}T_{2,\nu}T_{3,\nu^{-1}}.$$

THEOREM.—*The group H' in the $GF[p^n]$ has the order* *

$$\Omega \equiv p^{6n}(p^{6n}-1)(p^{2n}-1). \tag{34}$$

It has been shown that H' contains a transformation which replaces ξ_1 by

$$\sum_{j=0}^{3} a_{1j}\xi_j + \sum_{j=1}^{3} \gamma_{1j}\eta_j,$$

in which $a_{10}, \cdots, \gamma_{13}$ are arbitrary marks, not all zero, such that

$$\tfrac{1}{4}a_{10}^2 + a_{11}\gamma_{11} + a_{12}\gamma_{12} + a_{13}\gamma_{13} = 0. \tag{35}$$

By various methods, this equation is seen to have p^{6n} sets of solutions in the $GF[p^n]$. Hence $\Omega = p^{6n}\Omega_1$, where Ω_1 denotes the number of transformations of H' which leave ξ_1 fixed. As shown above, H' contains a transformation which leaves ξ_1 fixed and replaces η_1 by f_1, in which $\beta_{10}, \cdots, \delta_{13}$ are any marks which satisfy (15), viz.;

$$\tfrac{1}{4}\beta_{10}^2 + \beta_{11} + \beta_{12}\delta_{12} + \beta_{13}\delta_{13} = 0. \tag{36}$$

Hence $\beta_{10}, \beta_{12}, \delta_{12}, \beta_{13}, \delta_{13}$ may be chosen arbitrarily, the value of β_{11} being then determined uniquely. Hence $\Omega_1 = p^{5n}\Omega_2$, where Ω_2 denotes the number of transformations S_1 of H' which leaves ξ_1 and η_1 unaltered. As shown above,

* As a check, we note that p^n, the order of the field, enters to the power 14, which is the number of parameters in the continuous group.

S_1 must be of the form (33). Inversely, all the $p^n(p^{2n}-1)$ transformations (33) belong to H'. Hence the order of H' is given by (34).

We observe the special values:

$$p^n = 3, \quad \Omega = 2^6 \cdot 3^6 \cdot 7 \cdot 13 \equiv 4{,}245{,}696;$$

$$p^n = 5, \quad \Omega = 2^6 \cdot 3^3 \cdot 5^6 \cdot 7 \cdot 31 \equiv 5{,}859{,}000{,}000.$$

These simple groups are not in the writer's list of known simple groups.*

§ 10. *Simplicity of the group H'.*

Suppose that H' contains an invariant subgroup J which possesses a transformation S not the identity I. Let S replace ξ_1 by

$$f_1 \equiv a_{10}\xi_0 + \sum_{j=1}^{3}(a_{1j}\xi_j + \gamma_{1j}\eta_j) \qquad \text{[subject to (35)]}.$$

LEMMA: *J contains a transformation $\neq I$ which multiplies ξ_1 by a constant.*

(*a*) Let first $\gamma_{11} \neq 0$. It was shown in § 9 that H' contains a transformation R which replaces ξ_1 by ξ_1 and η_1 by

$$\beta_{10}\xi_0 + \beta_{11}\xi_1 + \eta_1 + \beta_{12}\xi_2 + \delta_{12}\eta_2 + \beta_{13}\xi_3 + \delta_{13}\eta_3,$$

in which β_{1i}, δ_{1i} are any quantities of the field F satisfying (36). By suitable choice of these quantities, the product

$$P \equiv T_{1,\gamma_{11}^{-1}} T_{2,\gamma_{11}} R$$

replaces ξ_1 by $\gamma_{11}^{-1}\xi_1$ and η_1 by f_1. Hence J contains

$$S_1 \equiv P^{-1}SP$$

which replaces ξ_1 by $\gamma_{11}^{-1}{}_1\eta_1$. If H' contains a transformation B leaving ξ_1 and η_1 unaltered and not commutative with S_1, then J contains $S_1^{-1}B^{-1}S_1B$, which leaves ξ_1 fixed and is not the identity. In the contrary case S_1 is commutative with $Q_{3,2,\lambda}$, $T_{2,-1}P_{23}$. Equating the functions by which $S_1Q_{3,2,\lambda}$ and $Q_{3,2,\lambda}S_1$ replace η_2 and the functions by which they replace ξ_3, we see that S_1 must replace ξ_2 and η_3 by respectively

$$\xi_2' = a\xi_2 + b\eta_3, \quad \eta_3' = c\eta_3 - d\xi_2.$$

Since S_1 is to be commutative with $T_{2,-1}P_{23}$, it replaces η_2 and ξ_3 by

$$\eta_2' = c\eta_2 + d\xi_3, \quad \xi_3' = a\xi_3 - b\eta_2.$$

Then (23), for $l=1$, $i=1$, gives $c=0$. Then J contains

$$S_1^{-1}(T_{1,\mu}T_{2,\mu-1})^{-1}S_1(T_{1,\mu}T_{2,\mu-1})$$

* Bulletin of the American Mathematical Society, ser. 2, vol. 5 (1899), p. 470–475.

which replaces η_2 by $\mu\eta_2$. Its transform by $T_{1,-1}P_{12}\Sigma$ replaces ξ_1 by $\mu\xi_1$ and is not the identity if $\mu \neq 0, 1$, where Σ is defined by (30).

(*b*) Let $\gamma_{11} = 0$, but a_{12} and a_{13} not both zero. By an evident transformation within H', we may make $a_{12} = 1$. Transforming S by

$$Y_{0,1,\gamma_{13}} W_{2,3,\gamma_{13}} \cdot Q_{2,3,a_{13}},$$

we reach in J a transformation S' which replaces ξ_1 by

$$a_{10}\xi_0 + a_{11}\xi_1 + \xi_2 + \gamma_{12}\eta_2 \qquad (\tfrac{1}{4}a_{10}^2 + \gamma_{12} = 0).$$

Then the transform of S' by $Q_{2,1,a_{11}}$ replaces ξ_1 by $\xi_2 + a_{10}\xi_0 - \frac{1}{4}a_{10}^2\eta_2$. Transforming by $X_{0,2,\frac{1}{2}a_{10}} V_{3,1,\frac{1}{2}a_{10}}$, we reach in J a transformation S_2 which replaces ξ_1 by ξ_2. Then J contains

$$S_2^{-1} \cdot T_{2,-1}T_{3,-1}S_2T_{2,-1}T_{3,-1}, \tag{37}$$

which replaces ξ_1 by $-\xi_1$.

(*c*) Let $\gamma_{11} = a_{12} = a_{13} = 0$. Then $a_{10} = 0$ by (35). If $\gamma_{12} = \gamma_{13} = 0$, S replaces ξ_1 by $a_{11}\xi_1$. In the contrary case, S is conjugate with a transformation S' with $\gamma_{12} = 1$; it replaces ξ_1 by $a_{11}\xi_1 + \eta_2 + \gamma_{13}\eta_3$. The transform of S' by $Q_{3,2,-\gamma_{13}}$ replaces ξ_1 by $a_{11}\xi_1 + \eta_2$. Transforming it by $X_{0,3,-a_{11}} V_{1,2,-a_{11}}$, we obtain a transformation S_2 which replaces ξ_1 by η_2. Then J contains (37), which replaces ξ_1 by $-\xi_1$.

LEMMA. *The group J contains a transformation, not the identity, which does not alter ξ_1 or η_1.*

In view of the preceding lemma, J contains a transformation $S \neq I$ which replaces ξ_1 by $a\xi_1$ and η_1 by

$$\beta_{10}\xi_0 + \sum_{j=1}^{3}(\beta_{1j}\xi_j + \delta_{1j}\eta_j) \qquad [\text{subject to } (15);\ \delta_{11} = a^{-1}].$$

According as δ_{12}, δ_{13} are both zero or not both zero, β_{12}, β_{13} both zero or not both zero, we distinguish four cases. Transforming by one or more of the transformations $T_{2,-1}P_{23}$, $Q_{2,3,\lambda}$, $Q_{3,2,\lambda}$, $T_{2,\lambda}T_{3,\lambda-1}$, we obtain a transformation in J which replaces ξ_1 by $a\xi_1$ and η_1 by one of the four functions

$$f \equiv \beta_{10}\xi_0 + \beta_{11}\xi_1 + a^{-1}\eta_1, \quad f + \xi_2, \quad f + \beta_{12}\xi_2 + \eta_2, \quad f + \beta_{13}\xi_3 + \eta_2.$$

Let first S replace ξ_1 by f. By (15), $a^{-1}\beta_{11} + \frac{1}{4}\beta_{10}^2 = 0$. Hence

$$S = T_{1,a}T_{2,a^{-1}}Y_{0,1,\beta}W_{2,3,\beta}S_1 \qquad (\beta \equiv \tfrac{1}{2}\beta_{10}),$$

where S_1 leaves ξ_1 and η_1 fixed, belongs to H', and hence has the form (33). If $\beta = 0$, the lemma is proved when $a = 1$. For $\beta = 0$, $a \neq 1$, S_1 does not reduce to the identity I or to $T_{2,-1}T_{3,-1}$ and hence is not commutative with every (33). If S_1 be not commutative with Σ_1 of the form (33), then J contains

$$S^{-1}\Sigma_1^{-1}S\Sigma_1 \equiv S_1^{-1}\Sigma_1^{-1}S_1\Sigma_1 \neq I$$

which leaves ξ_1 and η_1 unaltered. Let next $\beta \neq 0$. Then S is transformed by $T_{2,-1}P_{23}$ into $S_2 \equiv T_{1,\alpha}T_{3,\alpha^{-1}}Y_{0,1,\beta}W_{2,3,\beta}S'$, where

$$S' \equiv P_{23}T_{2,-1}S_1T_{2,-1}P_{23}.$$

Hence J contains the transformation

$$S_2' \equiv T_{2,\alpha}T_{3,\alpha^{-1}}S^{-1}T_{2,\alpha^{-1}}T_{3,\alpha}S_2 \equiv T_{2,\alpha}T_{3,\alpha^{-1}}S_1^{-1}S',$$

which leaves ξ_1 and η_1 unaltered. If S_2' be the identity, $\nu = a\kappa$, $\mu = -a\lambda$ in (33). Again, $T_{1,\alpha^{-1}}T_{2,\alpha}$ transforms S into $S_3 \equiv Y_{0,1,\beta}W_{2,3,\beta}S_1T_{1,\alpha^{-1}}T_{2,\alpha}$. Then J contains the product

$$S_4 \equiv S_3^{-1}Q_{2,3,1}^{-1}S_3Q_{2,3,1} \equiv T_{2,\alpha^{-1}}'S_1^{-1}Q_{2,3,1}^{-1}S_1T_{2,\alpha}Q_{2,3,1},$$

since $Q_{2,3,\tau}$ and $W_{2,3,\beta}$ are commutative. Now S_4 leaves ξ_1 and η_1 fixed. If S_4 reduce to the identity, we find that $\lambda = 0$, $\kappa = a\nu$ in (33). Hence if S_2' and S_4 are both the identity, (33) becomes $T_{2,\kappa^{-1}}T_{3,\kappa}$. In this case J contains

$$T_{1,-1}T_{2,-1}S^{-1}T_{1,-1}T_{2,-1}S \equiv T_{2,\kappa}T_{3,\kappa^{-1}}Y_{0,1,2\beta}W_{2,3,2\beta}T_{2,\kappa^{-1}}T_{3,\kappa},$$

and therefore its transform by $T_{2,\kappa}T_{3,\kappa^{-1}}'$, giving $Y_{0,1,2\beta}W_{2,3,2\beta}$. Hence J contains every $Y_{0,1,\tau}W_{2,3,\tau}$ and $X_{0,1,\tau}V_{2,3,\tau}$. As in § 9, we derive $\Sigma' = \Sigma T_{2,-1}P_{23}$, where Σ is given by (30). Transforming Σ' by $P_{23}T_{2,-1}$, we reach $T_{2,-1}P_{23}\Sigma$. The product of the two gives $T_{2,-1}T_{3,-1}$, which leaves ξ_1 and η_1 unaltered and is not the identity.

The remaining three cases may be treated in a similar manner.

In view of the two lemmas, the group J contains a transformation, not the identity, which does not alter ξ_1 or η_1 and hence (§ 9) has the form (33). If it be $T_{2,-1}T_{3,-1}$, it is transformed into $Q_{1,2,2\lambda}T_{2,-1}T_{3,-1}$ by $Q_{1,2,-\lambda}$. Hence would J contain $Q_{1,2,2\lambda}$ and therefore $Q_{2,3,2\lambda}$. It follows that J contains a transformation (33) neither I nor $T_{2,-1}T_{3,-1}$. But the transformations (33) form a group holoedrically isomorphic with the binary group $SLH(2, F)$. It follows that J contains all the transformations (33) and hence every $Q_{i,j,\tau}$, $T_{i,-1}P_{i,j}$, $T_{i,\mu}T_{j,\mu^{-1}}$. But $X_{0,3,\tau}V_{1,2,\tau}$ transforms $T_{2,-1}T_{3,-1}$ into $X_{0,3,-2\tau}V_{1,2,-2\tau}T_{2,-1}T_{3,-1}$. Hence J contains every $X_{0,3,\lambda}V_{1,2,\lambda}$. Transforming it by suitable $T_{i,-1}P_{i,j}$ and by Σ, we reach every $X_{0,i,\tau}V_{j,k,\tau}$, $Y_{0,i,\tau}W_{j,k,\tau}$. Hence $J \equiv H'$, so that H' is simple in any field F not having the modulus $p = 2$.

§ 11. *Linear groups with invariants of degree $d > 2$.*

Consider the group $G(q, r, F)$ of linear transformations S on rq variables with coefficients in an arbitrary field F which leave formally and absolutely invariant the function

$$\sum_{i=1}^{r} \xi_{i1}\xi_{i2}\cdots\xi_{iq}.$$

For $q > 2$, we may express every S as a product, $S = AB$, where A merely multiplies each variable ξ_{ij} by a constant a_{ij}, while B is a substitution on the letters ξ_{ij} having the imprimitive systems*

$$\xi_{11}, \xi_{12}, \cdots, \xi_{1q}; \quad \xi_{21}, \xi_{22}, \cdots, \xi_{2q}; \quad \cdots; \quad \xi_{r1}, \xi_{r2}, \cdots, \xi_{rq}.$$

The transformations A form a commutative group invariant under $G(q, r, F)$. The quotient-group $\{B\}$ has an invariant subgroup R, the direct product of r symmetric groups, the general one being the symmetric group on the q letters $\xi_{i1}, \xi_{i2}, \cdots, \xi_{iq}$. The quotient-group $\{B\}/R$ is the symmetric group on r letters, the above imprimitive systems.

Consider the group $H(m, r, F)$ of linear transformations S on m variables with coefficients in a field F, not possessing a modulus, which leave formally and absolutely invariant the function

$$\lambda_1 \xi_1^r + \lambda_2 \xi_2^r + \cdots + \lambda_m \xi_m^r \qquad (\text{each } \lambda_i \neq 0 \text{ in } F).$$

If $r > 2$, we may set $S = AL$, where A is a transformation of the form

$$\xi_i' = a_i \xi_i \quad [a_i^r = 1] \qquad (i = 1 \cdots, m),$$

and L is a literal substitution on the letters $\xi_1, \xi_2, \cdots, \xi_m$.†

On the other hand, there exist linear groups in an arbitrary field F which possess invariants of degree $d > 2$ and which lead to simple groups. Examples of such groups are furnished by the second compounds of the groups $GLH(m, F)$ and $GA(2m, F)$, each possessing an invariant Pfaffian (see § 6).

§ 12. *Canonical forms of linear homogeneous transformations.*

Consider a transformation with coefficients in a field F,

$$S: \qquad \xi_i' = a_{i1}\xi_1 + a_{i2}\xi_2 + \cdots + a_{im}\xi_m \qquad (i = 1, 2, \cdots, m).$$

The determination of a linear function which S multiplies by a constant K depends upon the *characteristic equation*

$$\Delta(K) \equiv \begin{vmatrix} a_{11} - K & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} - K & \cdots & a_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mm} - K \end{vmatrix} = 0.$$

If we introduce new variables defined by the transformation

$$T: \qquad \eta_i \equiv \beta_{i1}\xi_1 + \beta_{i2}\xi_2 + \cdots + \beta_{im}\xi_m \qquad (i = 1, 2, \cdots, m),$$

*Proceedings of the London Mathematical Society, vol. 30 (1899), pp. 200-208. The factor C should be $t_1!\, t_2! \cdots t_\tau!$ For an elementary treatment of the case $q = 3$, see L. G., §§ 211-212.

†Mathematische Annalen, vol. 52, p. 563; L. G., §§ 139-141.

the transformation S becomes the transformation $S_1 \equiv T^{-1}ST$ on the variables η_i and the characteristic determinant $\Delta(K)$ of S_1 is equal to the characteristic determinant of S (L. G., §§ 101, 102). *The characteristic equation is unaltered under linear transformation.*

Let $\Delta(K)$ be decomposed in the field F into irreducible factors,

$$\Delta(K) \equiv [F_k(K)]^\alpha [F_l(K)]^\beta \cdots \qquad (k\alpha + l\beta + \cdots = m).$$

Let $F_k(K) = 0$ have the roots $K_1, K_2, \cdots, K_k$; $F_l(K) = 0$ the roots $L_1, L_2, \cdots, L_l$; etc. To exhibit compactly the general type of canonical form of transformations S, let $\alpha, \beta, \cdots$ be partitioned into positive integers,

$$\alpha = a_1 + a_2 + \cdots + a_{r+1}, \quad \beta = b_1 + b_2 + \cdots + b_{s+1}, \quad \cdots$$

Let $a, b, \cdots$, denote an arbitrary one of the respective sets of integers

$$(a) \qquad 1, \quad a_1 + 1, \quad a_1 + a_2 + 1, \quad \cdots, \quad a_1 + a_2 + \cdots + a_r + 1;$$

$$(b) \qquad 1, \quad b_1 + 1, \quad b_1 + b_2 + 1, \quad \cdots, \quad b_1 + b_2 + \cdots + b_s + 1;$$

etc. Let A denote an arbitrary positive integer $\equiv a$ and not an a; let B denote an arbitrary positive integer $\equiv \beta$ and not a b; etc.

Proceeding as in L. G., §§ 214–218, we obtain the theorems:

By a suitable linear homogeneous transformation of variables (not belonging to F in general), S can be reduced to a canonical form

$$\eta'_{ia} = K_i \eta_{ia}, \quad \eta'_{iA} = K_i \eta_{iA} + K_i \eta_{iA-1} \qquad (i = 1, 2, \cdots, k)$$

$$\zeta'_{ib} = L_i \zeta_{ib}, \quad \zeta'_{iB} = L_i \zeta_{iB} + L_i \zeta_{iB-1} \qquad (i = 1, 2, \cdots, l)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

in which the new variables $\eta_{ij}, \zeta_{ij}, \cdots$ have the properties:

(1) *The variables $\eta_{1j} (j = 1, 2, \cdots, a)$ are linear homogeneous functions of the ξ_i whose coefficients are polynomials in K_1 with coefficients in F;*

(2) *The variables η_{ij} are obtained from the η_{1j} by replacing K_1 by K_i;*

(3) *The variables $\zeta_{1j} (j = 1, 2, \cdots, \beta)$ are linear functions of the ξ_i whose coefficients are polynomials in L_1 with coefficients in F;*

(4) *The variables ζ_{ij} are obtained from the ζ_{1j} by replacing L_1 by L_i;*

(5) *The $k\alpha$ variables $\eta_{ij} (i = 1, \cdots, k; j = 1, \cdots, a)$ may be replaced by $k\alpha$ linear homogeneous functions y_{ij} of the ξ_i with coefficients in F, such that S replaces each y_{ij} by a linear function of the y_{ij} with coefficients in F; similarly for the $l\beta$ variables ζ_{ij}; etc.*

Two linear homogeneous transformations S_1 and S_2 belonging to a field F have the same canonical form if, and only if, S_1 is the transform of S_2 by a linear homogeneous transformation T in the field F and on the same variables.

To determine all linear homogeneous transformations T commutative with a given one S, each in the field F and affecting m variables ξ_i, we apply the transformation of indices which reduces S to its canonical form S',

$$S' \equiv Y_1 Y_2 \cdots Y_k Z_1 Z_2 \cdots Z_l \cdots,$$

where each transformation Y_i, Z_i, $\cdots$, is defined thus:

$$Y_i: \quad \eta'_{ia} = K_i \eta_{ia}, \quad \eta'_{iA} = K_i \eta_{iA} + K_i \eta_{iA-1} \qquad \text{(for every } a, A);$$

$$Z_i: \quad \zeta'_{ib} = L_i \zeta_{ib}, \quad \zeta'_{iB} = L_i \zeta_{iB} + L_i \zeta_{iB-1} \qquad \text{(for every } b, B).$$

If T' be commutative with S', then

$$T' \equiv Y'_1 Y'_2 \cdots Y'_k Z'_1 Z'_2 \cdots Z'_l \cdots,$$

where Y'_i, Z'_i, $\cdots$, are of the form

$$Y'_i: \quad \eta'_{ij} = \sum_{u=1}^{a} \rho_{ju}(K_i) \eta_{iu} \qquad (j = 1, \cdots, a);$$

$$Z'_i: \quad \zeta'_{ij} = \sum_{v=1}^{\beta} \sigma_{jv}(L_i) \zeta_{iv} \qquad (j = 1, \cdots, \beta),$$

the coefficients of the polynomials $\rho_{ju}(K_i)$, $\cdots$, belonging to F.

Inversely, if T' have the above form and if Y'_1 be commutative with Y_1, Z'_1 commutative with Z_1, $\cdots$, then the transformation T (T' expressed in the initial variables ξ_i) will be commutative with S and will have its coefficients in the field F.

Y'_1 and Y_1 are commutative if, and only if, for every a, A, A',

$$\rho_{aA} = 0, \quad \rho_{A-1\,a-1} = 0, \quad \rho_{A-1\,a} = 0, \quad \rho_{A-1\,A'-1} = \rho_{AA'}.$$

THE UNIVERSITY OF CHICAGO,
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ON CERTAIN AGGREGATES OF DETERMINANT MINORS*

BY

W. H. METZLER

Introduction and General Notations.

1. Since the announcement † by KRONECKER, in 1882, of his now well-known theorem regarding linear relations between the minors of an axisymmetric determinant various papers ‡ have appeared treating of the subject. Dr. MUIR in his paper of 1888 showed that a similar relation exists between the minors of a centrosymmetric determinant and in his paper of 1900 he gives the following two theorems:

THEOREM A: *If μ and ν be any integers, μ being the less, taken from the series $n, n+1, n+2, \cdots, 2n$ and $\alpha, \beta, \gamma, \cdots, \omega$ be what the series becomes when μ is removed, and $\alpha, \beta, \gamma, \cdots, \psi$ what it becomes when both are removed; then in connection with any even-ordered determinant $\left|\begin{smallmatrix}1&2&\cdots&2n\\1&2&\cdots&2n\end{smallmatrix}\right|$ we have*

$$\sum_{\mu}(-1)^{\mu-\mu}\left|\begin{matrix}1\,2\,3\cdots\overline{n-1}\;\mu\\ \alpha\beta\gamma\cdots\cdots\;\omega\end{matrix}\right| = \sum_{\mu,\nu}(-1)^{3n-(\mu+\nu)}\left|\begin{matrix}1\,2\,3\cdots\overline{n-1}\\ \alpha\beta\gamma\cdots\;\psi\end{matrix}\right|\binom{\mu}{\nu}\binom{\nu}{\mu}.$$

THEOREM B: *If the symbol*

$$\sum_{\mu=1}^{n}\left|\begin{matrix}\overline{n+1}\;\overline{n+2}\cdots\overline{n-\mu+1}\cdots 2n\\ \overline{n+1}\;\overline{n+2}\cdots\;\overline{n+\mu}\;\cdots 2n\end{matrix}\right|$$

stand for the sum of the n determinants whose column-indices are in every case

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† KRONECKER, L., *Die Subdeterminanten symmetricher Systeme*, Berliner Berichte, 1882.

‡ RUNGE, C., *Die linearen Relationen zwischen den verschiedenen Subdeterminanten symmetricher Systeme*, Crelle's Journal, vol. 93.

MEHMKE, R., *Bemerkung über die Subdeterminanten symmetricher Systeme*, Mathematische Annalen, vol. 26.

SCHENDEL, L., *Die Kronecker'sche Subdeterminantensatz*, Zeitschrift für Mathematik und Physik, vol. 32.

MUIR, T., *On Vanishing Aggregates of Determinants*, Proceedings of the Royal Society of Edinburgh, 1888. *On Certain Aggregates of Determinant Minors*, ibid., 1900.

WHITE, H. S., *Kronecker's Linear Relation among Minors of a Symmetric Determinant*, Bulletin of the American Mathematical Society, 1896.

$n+1, n+2, \cdots, 2n$ and whose row-indices are the same except that for one of them there has been substituted its defect from $2n+1$; and if

$$\begin{vmatrix} \overline{n+1}\ \overline{n+2} \cdots \overline{n-\mu+1}' \cdots 2n \\ \overline{n+1}\ \overline{n+2} \cdots \quad \overline{n+\mu} \quad \cdots 2n \end{vmatrix}$$

be taken to indicate that in the determinant

$$\begin{vmatrix} \overline{n+1}\ \overline{n+2} \cdots \overline{n-\mu+1} \cdots 2n \\ \overline{n+1}\ \overline{n+2} \cdots \quad \overline{n+\mu} \quad \cdots 2n \end{vmatrix}$$

each element $\binom{\alpha}{\beta}$ of the μth row is to be diminished by the element $\binom{2n+1-\alpha}{2n+1-\beta}$; then,

$$\sum_{\mu=1}^{n} \left\{ \begin{vmatrix} \overline{n+1}\ \overline{n+2} \cdots \overline{n-\mu+1} \cdots 2n \\ \overline{n+1}\ \overline{n+2} \cdots \quad \overline{n+\mu} \quad \cdots 2n \end{vmatrix} - \begin{vmatrix} \overline{n+1}\ \overline{n+2} \cdots \quad \overline{n+\mu} \quad \cdots 2n \\ \overline{n+1}\ \overline{n+2} \cdots \overline{n-\mu+1} \cdots 2n \end{vmatrix} \right\}$$

$$= \sum_{\mu=1}^{n} \begin{vmatrix} \overline{n+1}\ \overline{n+2} \cdots \overline{n-\mu+1}' \cdots 2n \\ \overline{n+1}\ \overline{n+2} \cdots \quad \overline{n+\mu} \quad \cdots 2n \end{vmatrix}.$$

If the determinant $\left|\begin{smallmatrix} 1\,2 \cdots 2n \\ 1\,2 \cdots 2n \end{smallmatrix}\right|$ is axisymmetric then $\binom{\mu}{\nu} = \binom{\nu}{\mu}$ and the right-hand side of theorem A vanishes, giving KRONECKER'S relation. If the determinant is centrosymmetric then the right-hand side of theorem B vanishes, giving the relation pointed out by MUIR in 1888. In this paper three theorems [(1), (2), (3), below] and two formulas [I, II, below] are developed. Theorems A and B, which may be characterized by the notations $A(n, n-1)$ and $B(n, 1)$ respectively, are extended, giving theorems (1) and (3), characterized by the notations $A(n, k)$ and $B(n, k)$, $(1 \leqq k \leqq n-1)$, respectively. If we call the left-hand side of a Kronecker relation a Kronecker expression, then theorem (2) expresses a sum of Kronecker expressions as a sum of products of sums of Kronecker expressions of lower order. Formula I gives for a centrosymmetric determinant the number of true vanishing aggregates of each type, and formula II gives the number of those which are linearly independent. Part I (arts. 3–7) contains the work relating to theorem A and Part II (arts. 8–11) contains the work relating to theorem B.

2. If we are given any combination (selection) of $2n$ numbers k at a time, the combination of the remaining $2n-k$ numbers is said to be the complementary with respect to $2n$ of the given combination. Let us use here, as elsewhere,*

*Cf. American Journal of Mathematics, vol. 22, no. 1.

$(^{2n}_{\alpha}\ ^{k})$* to denote the αth combination k at a time of the $2n$ numbers $1, 2, \cdots, 2n$, $(^{2n\,|\,k}_{\alpha\ \beta}\ ^{l})$ to denote the βth combination l at a time of the numbers in the αth combination k at a time of the $2n$ numbers, and let $(^{\overline{2n}\,|\,k}_{\alpha})$ denote the complementary with respect to $2n$ of the combination $(^{2n}_{\alpha}\ ^{k})$; and similarly $(^{2n\,|\,\bar{k}}_{\alpha\ \beta}\ ^{l})$ will denote the complementary with respect to k of the combination $(^{2n\,|\,k}_{\alpha\ \beta}\ ^{l})$, i. e., the combination formed by the numbers remaining after the numbers in the combination $(^{2n\,|\,k}_{\alpha\ \beta}\ ^{l})$ are taken from the combination $(^{2n}_{\alpha}\ ^{k})$. The symbol $(^{\overline{2n}\,|\,\bar{k}}_{\alpha\ \beta}\ ^{l})$ will denote the complementary of $(^{2n\,|\,k}_{\alpha\ \beta}\ ^{l})$ with respect to k, i. e., the combination formed by the $2n-k-l$ numbers remaining after the l numbers in the combination $(^{2n\,|\,k}_{\alpha\ \beta}\ ^{l})$ are taken from the combination $(^{\overline{2n}}_{\alpha}\ ^{k})$. Similarly $(\overline{2n}\,|\bar{k}|\bar{l}\,|h\cdots)$ is the complementary of $(\overline{2n}\,|\bar{k}|\,l\,|h\cdots)$ with respect to l.

If (a), (b) denote combinations of distinct numbers the notations $(a\,\{\!|\,b)$ and $(a)\,(b)$ denote the combination of all the numbers involved in (a) and (b) taken respectively *in loco* or in natural order according as the brackets are or are not interlocked.

The notation s_t will be used for $s!\,t!\,(s-t)!$

Part I.

3. In this part of the paper expressions occur containing minors and products of minors of the following form: $\left|{}^{(x_1)}_{(x_2)}\right|$, $\left|{}^{(y_1)}_{(y_2)}\right| \cdot \left|{}^{(z_1)}_{(z_2)}\right|$, where $(x_1\,\{\!|\,x_2)$ and $(y_1\,\{\!|\,z_1\,\{\!|\,y_2\,\{\!|\,z_2)$ are permutations of the $2n$ numbers $1, 2, \cdots, 2n$. In general the sign factor for any term $\left|{}^{(x_1)}_{(x_2)}\right| \cdot \left|{}^{(y_1)}_{(y_2)}\right| \cdot \left|{}^{(z_1)}_{(z_2)}\right| \cdots$ is $(-1)^{\nu}$ where ν is the number of inversions of order in $(x_1\,\{\!|\,y_1\,\{\!|\,z_1)\cdots(x_2\,\{\!|\,y_2\,\{\!|\,z_2)\cdots$.

With respect to a parent determinant of order $2n$ the sum of minors of order n

$$\sum_{i_1=1}^{(2n-k)_n} (-1)^{\nu_1} \begin{vmatrix} (2n\,|\,k\,\{\!|\,2\overline{n}\,|\,k\,|\,n-k) \\ (2\overline{n}\,|\,k\,|\,n-k) \end{vmatrix}_{\alpha\ \ \ \alpha\ \ \ i_1},$$

which has k rows the same throughout may obviously be written in the form

$$\frac{1}{n_k} \sum_{i_1=1}^{(2n-k)_k} \sum_{i_2=1}^{(2n-2k)_{n-k}} (-1)^{\mu} \begin{vmatrix} (2n\,|\,k\,\{\!|\,2\overline{n}\,|\,k\,|\,k\,|\,n-k) \\ (2\overline{n}\,|\,k\,|\,k\,\{\!|\,2\overline{n}\,|\,k\,|\,k\,|\,n-k) \end{vmatrix}_{\alpha\ \ \ \alpha\ \ \ i_1\ \ \ i_2}$$

which, on expanding each term by Laplace's theorem in terms of minors formed from the αth selection of k rows with their complementaries, and finally collecting all the terms containing

$$\begin{vmatrix} (2n\,|\,k) \\ (2\overline{n}\,|\,k\,|\,k) \end{vmatrix}_{\alpha\ \ \ i_1}$$

* Unless otherwise stated, it is understood that the numbers in a combination are arranged in their natural order (order of magnitude).

as a factor, takes the form

$$\sum_{i_1=1}^{(2n-k)_k}(-1)^{\lambda}\begin{vmatrix}(2n\,|\,k)\\ \alpha\\ (2n\,|\,k\,|\,k)\\ \alpha\ i_1\end{vmatrix}\sum_{i_2=1}^{(2n-2k)_{n-k}}\begin{vmatrix}(\overline{2n}\,|\,\overline{k}\,|\,k\,|\,n-k)\\ \alpha\ i_1\ i_2\\ (\overline{2n}\,|\,\overline{k}\,|\,k\,|\,n-k)\\ \alpha\ i_1\ i_2\end{vmatrix}.$$

On examining the inner sum of this form one will easily see that any given selection of h, $1 \leqq h \leqq n-k$, of the $2n-2k$ numbers in the combination $(^{2n|\bar{k}|k}_{\alpha\ i_1})$ is constant as rows running through $(2n-2k-h)_{n-k}$ of the terms and constant as columns through as many more terms. We therefore arrive at the following:

THEOREM (1) or $A(n, k)$:

(1)

$$\sum_{i_1=1}^{(2n-k)_n}(-1)^{\nu_1}\begin{vmatrix}(2n\,|\,k\,\}\,\overline{2n}\,|\,k\,|\,n-k)\\ \alpha\qquad\alpha\qquad i_1\\ (\overline{2n}\,|\,\overline{k}\,|\,n-k)\\ \alpha\ i_1\end{vmatrix}=\sum_{i_1=1}^{(2n-k)_k}(-1)^{\nu_2}\begin{vmatrix}(2n\,|\,k)\\ \alpha\\ (\overline{2n}\,|\,k\,|\,k)\\ \alpha\ i_1\end{vmatrix}\frac{1}{2(n-k)_h}\sum_{\beta=1}^{(2n-2k)_h}S_\beta,$$

where S_β denotes the sum of the two aggregates:

$$\sum_{i_2=1}^{(2n-2k-h)_{n-k}}(-1)^{\nu_3}\begin{vmatrix}(\overline{2n}\,|\,k\,|\,\overline{k}\,|\,h\,\}\,\overline{2n}\,|\,k\,|\,\overline{k}\,|\,h\,|\,n-k-h)\\ \alpha\ i_1\ \beta\qquad\alpha\ i_1\ \beta\ i_2\\ (\overline{2n}\,|\,k\,|\,\overline{k}\,|\,h\,|\,n-k-h)\\ \alpha\ i_1\ \beta\ i_2\end{vmatrix},$$

$$\sum_{i_2=1}^{(2n-2k-h)_{n-k}}(-1)^{\nu_4}\begin{vmatrix}(2n\,|\,k\,|\,\overline{k}\,|\,h\,|\,n-k-h)\\ \alpha\ i_1\ \beta\ i_2\\ (\overline{2n}\,|\,k\,|\,\overline{k}\,|\,h\,\}\,\overline{2n}\,|\,k\,|\,\overline{k}\,|\,h\,|\,n-k-h)\\ \alpha\ i_1\ \beta\qquad\alpha\ i_1\ \beta\ i_2\end{vmatrix},$$

in which, if

$$(2n\,|\,k),\quad(\overline{2n}\,|\,\overline{k}\,|\,k\,|\,h),\quad(\overline{2n}\,|\,k\,|\,\overline{k}\,|\,h\,|\,n-k-h),\quad(\overline{2n}\,|\,k\,|\,k),\quad(\overline{2n}\,|\,k\,|\,k\,|\,h\,|\,n-k-h)$$

are denoted by

$$(a),\ (b),\ (c),\ (d),\ (e)$$

respectively, the integers ν_2, ν_3, ν_4 denote the numbers of inversions in

$$(a\,\}\,b\,\}\,c\,\}\,d\,\}\,e),\quad(b\,\}\,c\,\}\,e),\quad(e\,\}\,b\,\}\,c)$$

respectively.

The identity of this theorem (1) or $A(n, k)$ depends upon any combination $(^{2n|k}_{\alpha})$ and any integer h $(1 \leqq h \leqq n-k)$.

For $h = 1$ and $k = n-1$ theorem (1) reduces to MUIR's theorem A.

As an illustration of (1) we have for $(n, k, h) = (4, 2, 1)$:

$$\begin{vmatrix}1234\\5678\end{vmatrix} - \begin{vmatrix}1235\\4678\end{vmatrix} + \begin{vmatrix}1236\\4578\end{vmatrix} - \begin{vmatrix}1237\\4568\end{vmatrix} + \begin{vmatrix}1238\\4567\end{vmatrix} + \begin{vmatrix}1245\\3678\end{vmatrix} - \begin{vmatrix}1246\\3578\end{vmatrix} + \begin{vmatrix}1247\\3568\end{vmatrix}$$

$$- \begin{vmatrix}1248\\3567\end{vmatrix} + \begin{vmatrix}1256\\3478\end{vmatrix} - \begin{vmatrix}1257\\3468\end{vmatrix} + \begin{vmatrix}1258\\3467\end{vmatrix} + \begin{vmatrix}1267\\3458\end{vmatrix} - \begin{vmatrix}1268\\3457\end{vmatrix} + \begin{vmatrix}1278\\3456\end{vmatrix}$$

$$= \begin{vmatrix}12\\34\end{vmatrix} \left\{ \begin{vmatrix}56\\78\end{vmatrix} - \begin{vmatrix}57\\68\end{vmatrix} + \begin{vmatrix}58\\67\end{vmatrix} + \begin{vmatrix}67\\58\end{vmatrix} - \begin{vmatrix}68\\57\end{vmatrix} + \begin{vmatrix}78\\56\end{vmatrix} \right\}$$

$$- \begin{vmatrix}12\\35\end{vmatrix} \left\{ \begin{vmatrix}46\\78\end{vmatrix} - \begin{vmatrix}47\\68\end{vmatrix} + \begin{vmatrix}48\\67\end{vmatrix} + \begin{vmatrix}67\\48\end{vmatrix} - \begin{vmatrix}68\\47\end{vmatrix} + \begin{vmatrix}78\\46\end{vmatrix} \right\}$$

$$+ \text{etc.}$$

$$+ \begin{vmatrix}12\\78\end{vmatrix} \left\{ \begin{vmatrix}34\\56\end{vmatrix} - \begin{vmatrix}35\\46\end{vmatrix} + \begin{vmatrix}36\\45\end{vmatrix} + \begin{vmatrix}45\\36\end{vmatrix} - \begin{vmatrix}46\\35\end{vmatrix} + \begin{vmatrix}56\\34\end{vmatrix} \right\}.$$

4. The conjugate of every minor in the first aggregate of S_β in (1) is found in the second with the same or opposite sign according as $n - k$ is even or odd. This is readily seen on observing that in bringing $(b)(c)(e)$ to the form $(e)(b)(c)$ the $n - k$ numbers in (e) are carried over the $n - k$ numbers in (c) and (b) and therefore the exponents of the sign factors for the two differ, mod. 2, by $(n - k)^2$ or $n - k$.*

5. Since each aggregate of S_β on the right of (1) is an expression involving minors of order $n - k$ the same in form as those on the left-hand side of the equation we may, if we choose, apply the theorem to them. If $n - k$ is not odd we may select h so that, after applying the theorem a second time, $n - k - h$ is odd and shall have within each of the resulting S_β's conjugate minors with opposite signs.

6. If now we take the sum

$$(n-k)_g \sum_{i_1=1}^{(2n-k)_n} (-1)^{\nu_1} \begin{vmatrix} (2n\,|k|_a\,2\bar{n}\,|k|_a\,n-k)_{i_1} \\ (2\bar{n}\,|k|_a\,n-k)_{i_1} \end{vmatrix}$$

and, if instead of expanding each minor in terms of minors formed from the αth selection of k rows as in art. 3, we expand in terms of minor formed from the βth selection of h of the αth selection of k rows and g others and finally collect like terms we get the following:

THEOREM (2):

$$(2)\qquad (n-k)_g \sum_{i_1=1}^{(2n-k)_n} (-1)^{\nu_1} \begin{vmatrix} (2n|k|_a\,2\bar{n}\,|k|_a\,n-k)_{i_1} \\ (2\bar{n}\,|k|_a\,n-k)_{i_1} \end{vmatrix} = \sum_{i_1=1}^{(2n-k)_{h+2g}} (-1)^{\nu_2} P_{i_1},$$

* *Cf.* American Journal of Mathematics, vol. 22, no. 1, art. 2, p. 56.

where P_{i_1} denotes the product of the two aggregates:

$$\sum_{i_2=1}^{(h+2g)_g}(-1)^{\nu_3}\begin{vmatrix}(2n\,|\underset{\alpha}{k}|\,\underset{\beta}{h}\,\}\,2\overline{n\,|\underset{\alpha}{k}|\,h+2g}\,|\underset{i_2}{g})\\(\overline{2n}\,|\underset{\alpha}{k}|\,\overline{\underset{i_1}{h+2g}}\,|\underset{i_2}{g})\end{vmatrix},$$

$$\sum_{i_3=1}^{(2n-2g-h-k)_{n-h-g}}(-1)^{\nu_4}\begin{vmatrix}(2n\,|\underset{\alpha}{k}|\,\underset{\beta}{h}\,\}\,2\overline{n\,|\underset{\alpha}{k}|\,h+2g}\,|n-k-g)\\(\overline{2n}\,|\underset{\alpha}{k}|\,\overline{\underset{i_1}{h+2g}}\,|n-k-g)\end{vmatrix}.$$

If $h=k$ and $g=1$, then (2) reduces to (1).

As an example of (2) for $(n, k, g, h)=(3, 1, 1, 1)$ we have

$$\begin{aligned}2\left\{\left|\begin{smallmatrix}123\\456\end{smallmatrix}\right|-\left|\begin{smallmatrix}124\\356\end{smallmatrix}\right|+\left|\begin{smallmatrix}125\\346\end{smallmatrix}\right|-\left|\begin{smallmatrix}126\\345\end{smallmatrix}\right|+\left|\begin{smallmatrix}134\\256\end{smallmatrix}\right|-\left|\begin{smallmatrix}135\\246\end{smallmatrix}\right|+\left|\begin{smallmatrix}136\\245\end{smallmatrix}\right|+\left|\begin{smallmatrix}145\\236\end{smallmatrix}\right|-\left|\begin{smallmatrix}146\\235\end{smallmatrix}\right|+\left|\begin{smallmatrix}156\\234\end{smallmatrix}\right|\right\}\\
=\binom{2}{3}\left\{\left|\begin{smallmatrix}14\\56\end{smallmatrix}\right|-\left|\begin{smallmatrix}15\\46\end{smallmatrix}\right|+\left|\begin{smallmatrix}16\\45\end{smallmatrix}\right|\right\}-\binom{2}{4}\left\{\left|\begin{smallmatrix}13\\56\end{smallmatrix}\right|-\left|\begin{smallmatrix}15\\36\end{smallmatrix}\right|+\left|\begin{smallmatrix}16\\35\end{smallmatrix}\right|\right\}\\
+\binom{2}{5}\left\{\left|\begin{smallmatrix}13\\46\end{smallmatrix}\right|-\left|\begin{smallmatrix}14\\36\end{smallmatrix}\right|+\left|\begin{smallmatrix}16\\34\end{smallmatrix}\right|\right\}-\binom{2}{6}\left\{\left|\begin{smallmatrix}13\\45\end{smallmatrix}\right|-\left|\begin{smallmatrix}14\\35\end{smallmatrix}\right|+\left|\begin{smallmatrix}15\\34\end{smallmatrix}\right|\right\}\\
+\binom{3}{4}\left\{\left|\begin{smallmatrix}12\\56\end{smallmatrix}\right|-\left|\begin{smallmatrix}15\\26\end{smallmatrix}\right|+\left|\begin{smallmatrix}16\\25\end{smallmatrix}\right|\right\}-\binom{3}{5}\left\{\left|\begin{smallmatrix}12\\46\end{smallmatrix}\right|-\left|\begin{smallmatrix}14\\26\end{smallmatrix}\right|+\left|\begin{smallmatrix}16\\24\end{smallmatrix}\right|\right\}\\
+\binom{3}{6}\left\{\left|\begin{smallmatrix}12\\45\end{smallmatrix}\right|-\left|\begin{smallmatrix}14\\25\end{smallmatrix}\right|+\left|\begin{smallmatrix}15\\24\end{smallmatrix}\right|\right\}+\binom{4}{5}\left\{\left|\begin{smallmatrix}12\\36\end{smallmatrix}\right|-\left|\begin{smallmatrix}13\\26\end{smallmatrix}\right|+\left|\begin{smallmatrix}16\\23\end{smallmatrix}\right|\right\}\\
-\binom{4}{6}\left\{\left|\begin{smallmatrix}12\\35\end{smallmatrix}\right|-\left|\begin{smallmatrix}13\\25\end{smallmatrix}\right|+\left|\begin{smallmatrix}15\\23\end{smallmatrix}\right|\right\}+\binom{5}{6}\left\{\left|\begin{smallmatrix}12\\34\end{smallmatrix}\right|-\left|\begin{smallmatrix}13\\24\end{smallmatrix}\right|+\left|\begin{smallmatrix}14\\23\end{smallmatrix}\right|\right\}.\end{aligned}$$

7. If now we impose the conditions for axisymmetry the right-hand side of (1) becomes zero and for $k=n-1$ is the general Kronecker relation. Under the same conditions of axisymmetry the left-hand side of (2), which in general we shall denote by $(n-k)_g K(n, k)$, is a sum of Kronecker expressions and the theorem (2) may be written schematically

$$(n-k)_g K(n, k)=\sum K(h+g, h)\,K(n-g-h, k-h).$$

If $g=1$, then $K(h+g, h)$ is a Kronecker expression and if in addition $k=n-2$, then $K(n-g-h, k-h)$ is also a Kronecker expression, so that under these conditions (2) expresses *a sum of Kronecker expressions for minors of order n as a sum of products of Kronecker expressions for minors of orders $h+g$ and $n-h-g$ respectively.* It is apparent that, by repeated applications of the theorem and by a proper choice of g, we may reduce both factors of each term on the right of (2) to Kronecker expressions.

If the original determinant is skew then the two aggregates of S_β in (1) are equal.

Part II.

8. If (m) denotes any combination of the $2n$ numbers $1, 2, \cdots, 2n$, then $(\overline{m})$ shall denote the combination formed by replacing every number in (m) by its defect from $2n+1$.

The extension of theorem B is now easily stated as follows:

THEOREM (3) or $B(n, k)$:

$$(3)\quad \sum_{i_1=1}^{n_k} \begin{vmatrix} (\overline{2n\,|n|\,k})(2n\,|\overline{n}|\,k) \\ (2n\,|\,n) \end{vmatrix} - \sum_{i_1=1}^{n_k} \begin{vmatrix} (2n\,|\,n) \\ (\overline{2n\,|n|\,k})(2n\,|\overline{n}|\,k) \end{vmatrix}$$

$$= \sum_{i_1=1}^{n_k} \sum_{i_2=1}^{n_k} (-1)^{\nu} \begin{vmatrix} (2n\,|\overline{n}|\,k) \\ (2n\,|\overline{n}|\,k) \end{vmatrix} \left\{ \begin{vmatrix} (\overline{2n\,|n|\,k}) \\ (2n\,|n|\,k) \end{vmatrix} - \begin{vmatrix} (2n\,|n|\,k) \\ (\overline{2n\,|n|\,k}) \end{vmatrix} \right\}$$

where ν is the number of inversions in $^{(2n\,|\overline{n}|\,k)(2n\,|n|\,k)}_{\quad a\quad i_1\quad\quad a\quad i_1}$ *plus the number in* $^{(2n\,|\overline{n}|\,k)(2n\,|n|\,k)}_{\quad a\quad i_2\quad\quad a\quad i_2}$.

The proof of this theorem consists in the expansion by LAPLACE'S theorem of each minor contained in the first sum on the left in terms of minors formed from the $n-k$ rows having for indices the numbers in $^{(2n\,|\overline{n}|\,k)}_{\quad a\quad i_1}$, with their complementaries; and of each minor in the second sum on the left in terms of minors from the $n-k$ columns having the same numbers for column indices, with their complementaries, and finally in the collection of the terms in pairs one from each sum. Thus the left-hand side of (3) is equal to

$$\sum_{i_1=1}^{i_1=n_k} \left\{ \sum_{i_2=1}^{i_2=n_k} (-1)^{\nu} \begin{vmatrix} (\overline{2n\,|n|\,k}) \\ (2n\,|n|\,k) \end{vmatrix} \cdot \begin{vmatrix} (2n\,|\overline{n}|\,k) \\ (2n\,|\overline{n}|\,k) \end{vmatrix} \right\}$$

$$- \sum_{i_1=1}^{i_1=n_k} \left\{ \sum_{i_2=1}^{i_2=n_k} (-1)^{\nu} \begin{vmatrix} (2n\,|n|\,k) \\ (\overline{2n\,|n|\,k}) \end{vmatrix} \cdot \begin{vmatrix} (2n\,|\overline{n}|\,k) \\ (2n\,|\overline{n}|\,k) \end{vmatrix} \right\},$$

which in turn is equal to the right-hand side.

When $k=1$ the theorem $B(n, k)$ reduces to MUIR'S theorem B.

9. Under the condition for centrosymmetry the right-hand side of (3) vanishes and we have the general linear relation connecting minors of order n of a centrosymmetric determinant of order $2n$. The type $k=1$ is that which was first pointed out by MUIR in 1888.

10. MUIR in his paper of 1900 states in connection with the identities of the type $k=1$ given by him; "that the number of such identities possible, when the centrosymmetry is complete and the parent determinant is of the $2n$th order,

is $C_{2n,\,n}$." It is to be observed, however, that we do not get a true vanishing aggregate for every selection of the $2n$ numbers n at a time. For if x and y are any two of the n numbers in any given selection, then whenever $x + y = 2n + 1$ some of the minors in the first sum have their conjugates in the second and cancel each other, while others will vanish identically. Those that remain, if any, form an aggregate which is the extensional of a vanishing aggregate of minors of lower order.

Thus for a determinant of order six if we take the selection 256 we get

$$\begin{vmatrix}251\\256\end{vmatrix} + \begin{vmatrix}226\\256\end{vmatrix} + \begin{vmatrix}556\\256\end{vmatrix} - \begin{vmatrix}256\\251\end{vmatrix} - \begin{vmatrix}256\\226\end{vmatrix} - \begin{vmatrix}256\\556\end{vmatrix},$$

in which the first and fourth are conjugate and therefore cancel, the others are indentically zero having identical rows or columns.

For a determinant of order eight, taking the selection 1678 we get

$$\begin{vmatrix}1671\\1678\end{vmatrix} + \begin{vmatrix}1628\\1678\end{vmatrix} + \begin{vmatrix}1378\\1678\end{vmatrix} + \begin{vmatrix}8678\\1678\end{vmatrix} - \begin{vmatrix}1678\\1671\end{vmatrix} - \begin{vmatrix}1678\\1628\end{vmatrix} - \begin{vmatrix}1678\\1378\end{vmatrix} - \begin{vmatrix}1678\\8678\end{vmatrix}$$

which reduces to the extensional of

$$\begin{vmatrix}62\\67\end{vmatrix} + \begin{vmatrix}37\\67\end{vmatrix} - \begin{vmatrix}67\\62\end{vmatrix} - \begin{vmatrix}67\\37\end{vmatrix}.$$

Neglecting these degenerate aggregates we have, if λ denotes the number of true vanishing aggregates of each type (*i. e.*, for each of the $n - 1$ values of k), for the determination of λ the following:

FORMULA I:

$$\text{(I)} \qquad \lambda = \tfrac{1}{2}\left[C_{2n,\,n} - \sum_{r=1}^{r=[n/2]} \frac{n(n-1)\cdots(n-2r+1)}{(r!)^2}\, 2^{n-2r} \right].$$

This we see by counting the number of combinations containing one, two, three, etc., pairs of numbers satisfying the equation $x + y = 2n + 1$, and subtracting the result from $C_{2n,\,n}$ and observing that for every relation

$$\sum_{i_1=1}^{n_k} \begin{vmatrix} (\overset{}{2n}\,|n|\,k)_{a\ i_1}\,(2n\,|\overline{n}|\,k)_{a\ i_1} \\ (2n|n)_{a} \end{vmatrix} - \sum_{i_1=1}^{n_k} \begin{vmatrix} (2n|n)_{a} \\ (\overset{}{2n}\,|n|\,k)_{a\ i_1}\,(2n\,|\overline{n}|\,k)_{a\ i_1} \end{vmatrix} = 0,$$

there is another

$$\sum_{i_1=1}^{n_k} \begin{vmatrix} (\overline{2n}\,|n|\,k)_{a\ i_1}\,(\overline{2n}\,|\overline{n}|\,k)_{a\ i_1} \\ (\overline{2n}\,|n)_{a} \end{vmatrix} - \sum_{i_1=1}^{n_k} \begin{vmatrix} (2n|n)_{a} \\ (\overline{2n}\,|n|\,k)_{a\ i_1}\,(\overline{2n}\,|\overline{n}|\,k)_{a\ i_1} \end{vmatrix} = 0,$$

which in virtue of the centrosymmetry of the determinant is not different from the first.

11. On examining a set of λ true vanishing aggregates of any type k it will be seen that:

(a) Each aggregate contains n_k terms together with the negative of their conjugates.

(b) Every minor $\left|\begin{smallmatrix}(a)\\(b)\end{smallmatrix}\right|$ enters two aggregates, in one with (b) as the selection of columns and in another with (a), (or (a)), as the selection of columns.

(c) There are $n_k\lambda$ independent minors ($\frac{1}{2}n_k\lambda$ with their conjugates).

(d) For every set of $\frac{1}{2}\lambda$ aggregates there is a set of $\frac{1}{2}\lambda$ minors (one in each aggregate) whose conjugates are found in the remaining $\frac{1}{2}\lambda$ aggregates.

(e) The sum of the λ aggregates vanishes.

It follows from (a) and (b) that every minor enters one aggregate with the positive and one with the negative sign, and that any two aggregates have either two or no minors in common.

It follows from (d) that the sum of no fewer than λ aggegates can vanish.

If we denote the number of linearly independent aggregates of each type by μ then we have for the determination of μ the following:

FORMULA II:

(II) $$\mu = \lambda - 1.$$

SYRACUSE UNIVERSITY, *March* 15, 1901.

UEBER DIE ANWENDUNG DER CAUCHY'SCHEN MULTIPLICATIONSREGEL AUF BEDINGT CONVERGENTE ODER DIVERGENTE REIHEN*

VON

ALFRED PRINGSHEIM

In einer vor nahezu zwanzig Jahren verfassten Arbeit *über die Multiplication bedingt convergirender Reihen* † habe ich unter anderem gezeigt, dass das nach der CAUCHY'schen Regel gebildete Product aus einer nur *bedingt convergenten* oder sogar *divergenten* und einer *unbedingt* convergenten Reihe unter Umständen *unbedingt* convergirt. ‡ Und ich habe ferner hervorgehoben, dass auch für das Product *zweier bedingt convergenter* Reihen die Möglichkeit eines solchen Verhaltens keineswegs ausgeschlossen erscheine, ohne freilich damals im Stande zu sein, das wirkliche Vorkommen dieses Falles durch bestimmte Beispiele erweisen zu können. § Seitdem habe ich in Folge vielfacher Beschäftigung mit der Reihenlehre längst erkannt, dass das fragliche Verhalten für ganz grosse Kategorien von Reihen geradezu *typisch* ist, habe aber bisher nicht Gelegenheit genommen, hierüber etwas zu veröffentlichen. Nachdem nun Herr CAJORI in Bd. 2 dieser Zeitschrift, p. 25 ff. jene Frage wieder aufgenommen und durch Construction gewisser Beispiele in bejahendem Sinne entschieden hat, möchte ich mir erlauben, auch meine eigenen, auf ganz anderer und allgemeinerer Grundlage beruhenden Betrachtungen hier mitzutheilen und daran noch einige weitere Bemerkungen über den vorliegenden Gegenstand anzuknüpfen.

§ 1.

Es sei A_ν $(\nu = 0, 1, 2, \cdots)$ eine unbegrenzte Folge reeller oder complexer Zahlen von der Beschaffenheit, dass $\sum|A_\nu|$ *divergent* und $\mathrm{L}\, A_\nu = 0$ ist. Die Potenzreihe $\sum A_\nu x^\nu$ besitzt alsdann den Convergenzradius $|x| = 1$ und kann

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† Mathematische Annalen, Bd. 21 (1883), p. 327.

‡ a. a. o., pp 357-359.

§ a. a. o., p. 332.

auf dem Convergenzkreise *keinesfalls unbedingt*, wohl aber durchweg oder theilweise noch *bedingt* convergiren. Angenommen, es sei a, wo $|a|=1$, eine *Divergenz*-Stelle für $\sum A_\nu x^\nu$. Ordnet man alsdann $(x-a)\sum A_\nu x^\nu$ wiederum nach Potenzen von x, also:

$$(x-a)\sum_{\nu=0}^{\infty} A_\nu x^\nu = -A_0 a + \sum_{\nu=1}^{\infty}(A_{\nu-1} - A_\nu a)x^\nu,$$

so geht diese neue Potenzreihe für $x=a$ in die folgende über:

$$-A_0 a + \sum_{\nu=1}^{\infty}(A_{\nu-1}a^\nu - A_\nu a^{\nu+1}) = -A_0 a + \lim_{n=\infty}(A_0 a - A_n a^{n+1}) = 0$$

d. h. sie *convergirt* für $x=a$ (gegen die Summe 0), während sie im übrigen durchaus *gleichzeitig* mit der ursprünglichen Reihe $\sum A_\nu x^\nu$ *convergirt* oder *divergirt*.

Es möge nun $\sum A_\nu x^\nu$ für $|x|=1$ noch *bedingt* convergiren mit *einziger* Ausnahme der auf dem Einheitskreise gelegenen Stellen $a_1, a_2, \cdots, a_n$. Bildet man alsdann die Potenzreihe:

$$(1^a) \qquad \mathfrak{P}_1(x, a) = \prod_{\kappa=1}^{n}(x-a_\kappa)\cdot\sum_{\nu=0}^{\infty} A_\nu x^\nu,$$

indem man wiederum das rechts stehende Product nach Potenzen von x ordnet, so wird nach dem eben gesagten diese Reihe auf dem Einheitskreise (der dann nicht einmal mehr der *Convergenz*kreis zu sein braucht) *ausnahmslos* convergiren. Daraus darf freilich noch nicht ohne weiteres geschlossen werden, dass die Convergenz dann allemal noch eine *unbedingte* (absolute) sein müsse.* Immerhin wird man sagen können, dass "*im allgemeinen*" oder, um diesen Ausdruck noch genauer zu praecisiren, in unendlich vielen, durch geeignete Auswahl der A_ν in beliebiger Anzahl leicht zu erzielenden Fällen $\mathfrak{P}_1(x, a)$ für $|x|=1$ *unbedingt* convergiren wird.

Sei ferner $\sum B_\nu x^\nu$ eine ganz ähnliche Potenzreihe, die für alle Stellen auf dem Einheitskreise mit einziger Ausnahme von $\beta_1, \beta_2, \cdots, \beta_n$ (wo $\beta_\lambda \neq a_\kappa$) noch *bedingt* convergirt, sodass also:

$$(2^a) \qquad \mathfrak{P}_2(x, \beta) = \prod_{\lambda=1}^{n}(x-\beta_\lambda)\cdot\sum_{\nu=0}^{\infty} B_\nu x^\nu$$

für $|x|=1$ widerum *ausnahmslos* und, bei geeigneter Auswahl der B_ν, auch *unbedingt* convergirt. Bildet man nun andererseits die beiden Potenzreihen:

$$(1^b) \qquad \mathfrak{P}_1(x, \beta) = \prod_{\lambda=1}^{n}(x-\beta_\lambda)\cdot\sum_{\nu=0}^{\infty} A_\nu x^\nu$$

*Mathematische Annalen, Bd. 25 (1885), p. 419. Münchener Berichte, Bd. 30 (1900), p. 71.

$$(2^b) \qquad \mathfrak{P}_2(x, a) = \prod_{\kappa=1}^{m}(x - a_\kappa) \cdot \sum_{\nu=0}^{\infty} B_\nu x^\nu,$$

so wird auf Grund der zu Anfang gemachten Bemerkung die *erstere* an den *Divergenzstellen* von $\sum A_\nu x^\nu$, also die Stellen a_κ und, entsprechend, die *zweite* an den Stellen β_λ noch *divergiren*, während im übrigen für $|x| = 1$ beide Reihen *bedingt convergiren*.

Nun ist aber offenbar:

$$(3) \qquad \mathfrak{P}_1(x, \beta) \cdot \mathfrak{P}_2(x, a) = \mathfrak{P}_1(x, a) \cdot \mathfrak{P}_2(x, \beta),$$

und diese *Gleichheit* wird zu einer vollkommenen *Identität*, wenn man ihre beiden Seiten wieder in Potenzreihen umformt, d. h. die CAUCHY'sche Multiplicationsregel darauf anwendet. Wegen der für $|x| = 1$ noch *unbedingten* Convergenz von $\mathfrak{P}_1(x, a)$, $\mathfrak{P}_2(x, \beta)$ besteht dann diese letztere auch für die *Productreihe*, zu deren Herstellung auch die für $|x| = 1$ nur *bedingt convergenten*, an den Stellen $x = a_\kappa$ bezw. $x = \beta_\lambda$ sogar *divergenten* Reihen $\mathfrak{P}_1(x, \beta)$, $\mathfrak{P}_2(x, a)$ dienen können.

Es hat keine Schwierigkeit, dieses Resultat auch auf den Fall zu übertragen, dass *eine* der beiden zu multipliciirenden Reihen *unbedingt* convergirt. Man hat dann nur $\sum B_\nu x^\nu$ als für $|x| = 1$ noch *unbedingt* convergent anzunehmen und etwa zu setzen:

$$(4) \qquad \mathfrak{P}_1(x) = \sum_{\nu=0}^{\infty} A_\nu x_\nu, \quad \mathfrak{P}_2(x) = \sum_{\nu=0}^{\infty} B_\nu x^\nu.$$

Behalten dann $\mathfrak{P}_1(x, a)$, $\mathfrak{P}_2(x, a)$ die in Gl. (1^a), (2^b) festgesetzte Bedeutung, so hat man wiederum:

$$(5) \qquad \mathfrak{P}_1(x) \cdot \mathfrak{P}_2(x, a) = \mathfrak{P}_1(x, a) \cdot \mathfrak{P}_2(x)$$

sodass $\mathfrak{P}_1(x)$, $\mathfrak{P}_2(x, a)$, obschon die *erste* dieser beiden Reihen für $|x| = 1$ nur *bedingt convergirt* bezw. für $x = a_\kappa$ *divergirt*, wegen der *unbedingten* Convergenz von $\mathfrak{P}_1(x, a) \cdot \mathfrak{P}_2(x)$ wieder eine *unbedingt* convergirende Productreihe liefert.

Hernach ergiebt sich:

Die Anwendung der Cauchy'schen Multiplicationsregel auf zwei Reihen, von denen die erste nur bedingt convergirt oder sogar divergirt, während die zweite bedingt oder unbedingt convergirt, liefert in unendlich vielen Fällen eine unbedingt convergirende Reihe.

§ 2.

Um jetzt Reihen der fraglichen Art mit lauten *reellen* Gliedern zu erhalten, wird man vor allem die A_ν, B_ν und x *reell*, also $x = \pm 1$ anzunehmen haben. Die a_κ bezw. β_λ müssen sodann gleichfalls *reell* oder *paarweise conjugirt com-*

plex gewählt werden. Um auf dem Einheitskreise wenigstens *eine reelle Convergenz*-Stelle, etwa $x=1$, zu behalten, bleibt dann als einzige mögliche *reelle Divergenz*-Stelle für $\sum A_\nu x^\nu$ die Stelle $x=-1$ (d. h. die α_κ reduciren sich auf den *einen* Werth $\alpha_1=-1$). Als einfachste Wahl für die *Divergenz*-Stellen von $\sum B_\nu x^\nu$ bleibt dann $\beta_1=i$, $\beta_2=-i$. Man genügt offenbar allen diesen Bedingungen, wenn man setzt:

$$A_\nu=(-1)^\nu a_\nu, \quad B_\nu=(-1)^{[\nu/2]} b_\nu,^*$$

wo die a_ν, b_ν positive, mit unbegrenzt wachsenden ν *monoton* gegen *Null* abnehmende Zahlen mit *divergenten* $\sum a_\nu$, $\sum b_\nu$ vorstellen. In der That sind dann die beiden Reihen:

$$\sum_{\nu=0}^{\infty}(-1)^\nu a_\nu x^\nu = a_0 - a_1 x + a_2 x^2 - a_3 x^3 + - + - \cdots$$

$$\sum_{\nu=0}^{\infty}(-1)^{[\nu/2]} b_\nu x^\nu = b_0 + b_1 x - b_2 x^2 - b_3 x^3 + + - - \cdots$$

für $|x|=1$ noch *bedingt convergent*, die *erste* mit einziger Ausnahme von $x=-1$, die *zweite* mit Ausnahme von $x=\pm i$. Bildet man ferner nach Analogie von Gl. (1^a), (2^a):

$$(5)\quad \begin{cases} (1+x)\sum_{\nu=0}^{\infty}(-1)^\nu a_\nu x^\nu = a_0 + \sum_{\nu=1}^{\infty}(a_{\nu-1}-a_\nu)x^\nu \\ (1+x^2)\sum_{\nu=0}^{\infty}(-1)^{[\nu/2]} b_\nu x^\nu = b_0 + b_1 x + \sum_{\nu=2}^{\infty}(-1)^{[\nu/2]-1}(b_{\nu-2}-b_\nu)x^\nu, \end{cases}$$

so erkennt man unmittelbar, dass diese Reihen für $|x|=1$, speciell also für $x=\pm 1$ noch *unbedingt* convergiren. Andererseits bleibt von den nach Analogie des Gl. (1^b), (3^b) gebildeten Reihen:

$$(6)\quad \begin{cases} (1+x^2)\sum_{\nu=0}^{\infty}(-1)^\nu a_\nu x^\nu = a_0 - a_1 x + \sum_{\nu=2}^{\infty}(-1)^\nu (a_{\nu-1}+a_\nu)x^\nu \\ (1+x)\sum_{\nu=0}^{\infty}(-1)^{[\nu/2]} b_\nu x^\nu = b_0 + \sum_{\nu=1}^{\infty}(-1)^{[(\nu-1)/2]}(b_{\nu-1}+(-1)^{\nu-1}b_\nu)x^\nu \end{cases}$$

die *erste* für $x=1$, die *zweite* für $x=\pm 1$ noch *bedingt convergent*, während die *erste* für $x=-1$ (die *zweite* nur für $x=\pm i$) *divergirt*.

Setzt man also einmal $x=+1$, sodann $x=-1$, so folgt, dass die Reihen:

$$(A)\quad \left\{a_0 - a_1 + \sum_{\nu=2}^{\infty}(-1)^\nu (a_{\nu-2}+a_\nu)\right\}\cdot\left\{b_0 + \sum_{\nu=1}^{\infty}(-1)^{[(\nu-1)/2]}(b_{\nu-1}+(-1)^{\nu-1}b_\nu)\right\}$$

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* Dabei bedeutet $[\nu/2]$ die grösste in $\nu/2$ enthaltene ganze Zahl.

$$(B)\quad \left\{a_0 + a_1 + \sum_{\nu=2}^{\infty}(a_{\nu-2} + a_\nu)\right\}\cdot\left\{b_0 + \sum_{\nu=1}^{\infty}(-1)^{\nu+[(\nu-1)/2]}(b_{\nu-1} + (-1)^{\nu-1}b_\nu)\right\}$$

unbedingt convergente Productreihen liefern, obschon die *beiden* Reihen (A) und die *zweite* (B) *nur bedingt convergiren*, während die *erste* (B) sogar *divergirt*. Die Ausführung der Multiplication nach der CAUCHY'schen Regel liefert nämlich genau dasselbe Resultat, wie für die aus (5) für $x = \pm 1$ resultirenden, *unbedingt* convergirenden Reihen:

$$(A')\quad \left\{a_0 + \sum_{\nu=1}^{\infty}(-1)^{\nu-1}(a_{\nu-1} - a_\nu)\right\}\cdot\left\{b_0 + b_1 + \sum_{\nu=2}^{\infty}(-1)^{[\nu/2]-1}(b_{\nu-2} - b_\nu)\right\}$$

$$(B')\quad \left\{a_0 - \sum_{\nu=1}^{\infty}(a_{\nu-1} - a_\nu)\right\}\cdot\left\{b_0 + b_1 - \sum_{\nu=2}^{\infty}(-1)^{\nu+[\nu/2]}(b_{\nu-2} - b_\nu)\right\}.$$

Will man in (A) und (B) an die Stelle des *zweiten* Factors eine *unbedingt* convergirende Reihe treten lassen, so hat man nur $\sum c_\nu$ als *unbedingt* convergent, im übrigen beliebig anzunehmen, und findet dann auf Grund der Beziehung:

$$\left\{\sum_{\nu=0}^{\infty}(-1)^\nu a_\nu x^\nu\right\}\cdot\left\{(1+x)\sum_{\nu=0}^{\infty}c_\nu x^\nu\right\} = \left\{(1+x)\sum_{\nu=0}^{\infty}(-1)^\nu a_\nu x^\nu\right\}\cdot\left\{\sum_{\nu=0}^{\infty}c_\nu x^\nu\right\},$$

dass die Reihen:

$$(C)\quad \left\{\sum_{\nu=0}^{\infty}(-1)^\nu a_\nu\right\}\cdot\left\{c_0 + \sum_{\nu=1}^{\infty}(c_{\nu-1} + c_\nu)\right\}$$

$$(D)\quad \left\{\sum_{\nu=0}^{\infty}a_\nu\right\}\cdot\left\{c_0 + \sum_{\nu=1}^{\infty}(-1)^\nu(c_{\nu-1} + c_\nu)\right\}$$

wiederum *unbedingt* convergirende Productreihen liefern, nämlich solche, die identisch sind mit denjenigen der *unbedingt* convergirende Factoren:

$$(C')\quad \left\{a_0 + \sum_{\nu=1}^{\infty}(-1)^{\nu-1}(a_{\nu-1} - a_\nu)\right\}\cdot\left\{\sum_{\nu=0}^{\infty}c_\nu\right\},$$

$$(D')\quad \left\{a_0 - \sum_{\nu=1}^{\infty}(a_{\nu-1} - a_\nu)\right\}\cdot\left\{\sum_{\nu=0}^{\infty}(-1)^\nu c_\nu\right\}.$$

Die auf diese Weise gewonnenen Beispiele können auch dazu dienen, um die Richtigkeit des am Schlusse von § 1 ausgesprochenen Resultates mit den *elementarsten* Mittel der Reihenlehre zu erweisen, d. h. *ohne* im übrigen von der Theorie der *Potenzreihen*, welche zur *Auffindung* dieser Beispiele gedient hatte, irgend welchen Gebrauch zu machen. Einerseits kann nämlich der Convergenz-

Charakter sämmtlicher in $(A)-(D)$, $(A')-(D')$ auftretenden Reihen ganz unmittelbar erkannt werden. Sodann liefert die Ausführung der Multiplication nach der CAUCHY'schen Regel für $(A)-(D)$ genau dieselben Productreihen, wie für $(A')-(D')$. Schliesslich resultirt dann deren *unbedingten* Convergenz aus derjenigen aller in $(A')-(D')$ vorkommenden Reihen.

Ferner bemerke man noch folgendes. Die *unbedingte* Convergenz der in (A'), (B') vorkommenden Reihen wird in keiner Weise alterirt, wenn man a_ν durch $a_\nu + a$ *oder* bezw. *und* b_ν durch $b_\nu + b$ ersetzt, wo a, b irgend zwei positive Zahlen bedeuten. Hierdurch werden aber andererseits die entsprechenden Reihen in (A), (B) durchweg *divergent*. Daraus folgt aber:

Auch die Multiplication zweier divergenter Reihen kann eine unbedingt convergirende Reihe liefern.

§ 3.

Die in den vorigen Paragraphen gewonnenen Resultate beruhten wesentlich darauf, dass man eine *bedingt convergente* oder *divergente* Reihe $\sum u_\nu$ als Specialfall von $\sum u_\nu x^\nu$ für $x = 1$ auffasste. (N. B. Die Voraussetzung, dass dabei $\sum u_\nu x^\nu$ den Convergenzradius 1 besitzen sollte, ist allemal *eo ipso* erfüllt, wenn $\sum u_\nu$ *bedingt convergirt;* wenn dagegen $\sum u_\nu$ *divergirt*, so muss noch ausdrücklich angenommen werden, dass $\mathop{\mathrm{L}}\sqrt[\nu]{|u_\nu|} = 1$). Es werde nun gesetzt:

$$\sum_{\nu=0}^{\infty} u_\nu x^\nu = f(x) \quad \text{für } |x| < 1,$$

und sodann $f(X)$ für alle Stellen X auf den Convergenzkreise $|X| = 1$ durch die Gleichung definirt:

$$f(X) = \mathop{\mathrm{L}}_{\rho=1-0} \sum_{\nu=0}^{\infty} u_\nu (\rho X)^\nu,$$

(sodass also nach dem bekannten ABEL'schen Grenzwerth-Satze:

$$f(X) = \sum_{\nu=0}^{\nu=\infty} u_\nu X^\nu,$$

sobald diese Reihe *convergirt*). Um unnöthige Complicationen zu vermeiden, mag angenommen werden, dass $f(x)$ auf dem Einheitskreise eine nur *endliche* Anzahl *gewöhnlichen* Unendlichkeits-Stellen (d. h. *ohne Oscillationen*) $x = a_\varkappa$ besitze, im übrigen längs des Einheitskreises *stetig* und zum mindesten in der Nähe von $x = 1$ *ohne Oscillationen* verlaufe. Die *Convergenz* oder *Divergenz* von $\sum u_\nu$ wird dann nach bekannten Sätzen aus der Theorie der FOURIER'schen Reihen einzig und allein von der *Art des Unendlichwerdens* von $f(x)$ für $x = a_\varkappa$ abhängen. Befindet sich unter den Stellen $a_\varkappa$ mindestens *eine* solche, für welche $f(x)$ von der *ersten* oder noch *höheren* Ordnung unendlich wird, so ist $\sum u_\nu$ alle-

mal *divergent*, da die u_ν in diesem Falle für unbegrenzt wachsende ν *nicht* gegen Null convergiren. Wohingegen $\sum u_\nu$ *convergiren* wird, wenn $f(x)$ an den Stellen $a_\varkappa$ durchweg von *niederer* oder, etwas genauer gesagt, von "*hinlänglich*" *niederer** Ordnung, als der *ersten* unendlich wird.

Es bedeute ferner $\sum v_\nu x^\nu$ eine Potenzreihe mit ganz analogen Eigenschaften, und es werde gesetzt:

$$\sum_{\nu=0}^{\infty} v_\nu x^\nu = g(x) \quad \text{für } |x| < 1,$$

$$g(X) = \mathop{\mathrm{L}}_{\rho=1-0} \sum_{\nu=0}^{\infty} v_\nu (\rho X)^\nu,$$

während die Unendlichkeits-Stellen von $g(X)$ mit β_λ bezeichnet werden mögen.

Da die Multiplication von $\sum u_\nu \cdot \sum v_\nu$ nach der CAUCHY'schen Regel genau dasselbe Resultate ergiebt, als wenn man das Product $\sum u_\nu x^\nu \cdot \sum v_\nu x^\nu$ zunächst nach Potenzen von x ordnet und sodann $x = 1$ setzt, so wird die *Convergenz* oder *Divergenz* jener ersten Productreihe lediglich davon abhängen, ob überhaupt bezw. in welcher Weise $f(x)$, $g(x)$ auf dem Convergenzkreise *unendlich* wird.

Sind die β_λ durchweg von den $a_\varkappa$ *verschieden*, so wird hiernach die Productreihe $\sum u_\nu \cdot \sum v_\nu$ sicher dann *convergiren*, wenn $\sum u_\nu$, $\sum v_\nu$ einzeln *convergirten*. Daraus erklärt es sich ohne weiteres, warum zwei Reihen, wie die zu Anfang von § 2 eingeführten, nämlich:

$$a_0 - a_1 + a_2 - a_3 + - + - \cdots \qquad \left(a_{\nu+1} \leqq a_\nu,\ \mathrm{L}\, a_\nu = 0\right)$$
$$b_0 + b_1 - b_2 - b_3 + + - - \cdots \qquad \left(b_{\nu+1} \leqq b_\nu,\ \mathrm{L}\, b_\nu = 0\right)$$

allemal ein *convergentes* Product liefern,† mögen die a_ν, b_ν auch *noch so langsam* gegen Null abnehmen.

Es können sich aber, immer unter der voraussetzung $\beta_\lambda \neq a_\varkappa$, die Convergenz-Chancen für die Productreihe auch erheblich *günstiger* gestalten, als für die Einzelreihen $\sum u_\nu$, $\sum v_\nu$, nämlich *dann*, wenn $\sum u_\nu x^\nu$ die β_λ, $\sum v_\nu x^\nu$ die $a_\varkappa$ in's gesammt oder doch theilweise zu *Nullstellen* hat. Eine Illustration hierzu liefern die Betrachtungen der beiden vorangehenden Paragraphen.

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* d. h. *so*, dass $f(x)$ bei $x = a_\varkappa$ *integrabel* bleibt, wie z. B.

$$\left(\frac{1}{x - a_\varkappa}\right)^{1-\varepsilon} \quad (0 < \varepsilon < 1), \qquad \frac{1}{(x-a_\varkappa)\,[lg(x-a_\varkappa)]^{1+\rho}} \quad (\rho < 0).$$

Wird $f(x)$ zwar von *niederer*, als der *ersten*, jedoch *nicht integrabel-unendlich*, wie z. B.

$$\frac{1}{(x-a_\varkappa)\,lg(x-a_\varkappa)},$$

so muss noch eine gewisse Bedingung hinzukommen, wenn $\mathrm{L} u^\nu = 0$ werden soll. Vgl. RIEMANN, Ges. W., pp. 244–245. Münchener Berichte, Bd. 25 (1895), pp. 339–362.

† Mathematische Annalen, Bd. 21 (1883), p. 346.

Hingegen tritt nun allemal eine *Verschlechterung* der Convergenz-Chancen für die Productreihe ein, sobald irgend ein β_λ mit einem a_κ *zusammenfällt*, da hierdurch für $f(x) \cdot g(x)$ eine Unendlichkeits-Stelle von entsprechend zusammengesetzter *höherer* Ordnung erzeugt wird: sobald diese Ordnung die *erste* erreicht oder übersteigt, muss dann die Productreihe allemal *divergiren*.

Aus dieser Bemerkung erklärt sich nun aber in sehr einfacher Weise die von Herrn CAJORI * hervorgehobene Thatsache, dass die Potenzen gewisser bedingt convergirender Reihen—wie $\sum_{\nu=1}^{\nu=\infty}(-1)^{\nu-1}\nu^{-r}$ $(0<r<1)$—*aufhören zu convergiren*, sobald der Exponent q eine bestimmte, von r abhängige Zahl erreicht oder übersteigt; während für andere Reihen—wie $\sum_{\nu=1}^{\nu=\infty}(-1)^{\nu-1}\nu^{-1}$ —jede noch so hohe Potenz eine *convergente* Reihe liefert. Von den beiden Reihen:

$$\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\frac{x^\nu}{\nu^r}, \quad \sum_{\nu=1}^{\infty}(-1)^{\nu-1}\frac{x^\nu}{\nu}$$

wird nämlich für $x=-1$ die *erste so* unendlich,† wie $-(1+x)^{-(1-r)}$, die *zweite*, wie $lg(1+x)$. Erhebt man also die *erste* Reihe in die q-te Potenz, so entsteht bei $x=-1$ eine Unendlichkeits-Stelle mit der Ordnungszahl $q(1-r)$, und es tritt daher *Divergenz* ein, sobald:

$$q(1-r) \geqq 1, \text{ d. h. } q \geqq \frac{1}{1-r}, \text{ anders geschrieben: } r \leqq 1-\frac{1}{q}.$$

Dagegen werden die Potenzen der *zweiten* Reihe für beliebig grosse Exponenten q bei $x=-1$ immer nur eine Unendlichkeits-Stelle vom Typus $[lg(1+x)]^q$ besitzen, sodass also die *Convergenz* erhalten bleibt.

§ 4.

Ich möchte diese Gelegenheit benützen, um eine Lücke auszufüllen, die mir bei der Durchsicht meiner zu Anfang dieser Note citirten Abhandlung ("*Ueber die Multiplication bedingt convergenter Reihen*") aufgefallen ist. Daselbst (a. a. o., p. 363) wird der folgende Satz ausgesprochen:

Sind a_n, b_n positive, mit wachsendem n niemals zunehmende und für $n=\infty$ verschwindende Zahlen, so bildet die Beziehung:

$$\mathop{\mathrm{L}}_{n=\infty} b_n \sum_{\kappa=1}^{n} a_\kappa = \mathop{\mathrm{L}}_{n=\infty} a_n \sum_{\kappa=1}^{n} b_\kappa = 0$$

die *nothwendige und hinreichende* Bedingung für Anwendbarkeit der Multiplicationsregel auf die bedingt convergenten Reihen:

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* A. a. O., p. 25. American Journal of Mathematics, vol. 18, p. 201.

† Nach einem Satze von P. APPELL: Comptes rendus, T. 87 (1878), p. 690. (Auch bei PICARD, *Traité d'Analyse*, T. I, p. 210).

$$\sum_{\nu=0}^{\infty}(-1)^{\nu}a_{\nu},\quad \sum_{\nu=0}^{\infty}(-1)^{\nu}b_{\nu}.$$

Der Beweis dafür, dass die obige Bedingung *hinreicht*, wird an der betreffenden Stelle richtig geführt. Dagegen ist der dort gegebene Beweis für deren *Nothwendigkeit* nicht nur von überflüssiger Umständlichkeit, sondern in Folge der Benützung gewisser aus den gemachten Voraussetzungen noch keineswegs folgenden Beziehungen, nämlich:

$$\mathop{\mathrm{L}}_{\nu=\infty}\frac{a_{\nu+1}}{a_{\nu}}=1,\quad \mathop{\mathrm{L}}_{\nu=\infty}\nu\left(1-\frac{a_{\nu+1}}{a_{\nu}}\right)>1,$$

sogar unzulänglich. In Wahrheit ergiebt sich aber die *Nothwendigkeit* der fraglichen Bedingung ganz unmittelbar in folgender Weise. Die Ausführung der Multiplication nach der CAUCHY'schen Regel liefert die Reihe:

$$\sum_{\nu=0}^{\infty}(-1)^{\nu}c_{\nu},\quad \text{wo:}\quad c_{n}=\sum_{\kappa=0}^{n}a_{\kappa}b_{n-\kappa}\begin{cases}>b_{n}\sum_{\kappa=0}^{n}a_{\kappa}\\ >a_{n}\sum_{\kappa=0}^{n}b_{\kappa}.\end{cases}$$

Da aber die Bedingung $\mathrm{L}\,c_{n}=0$ eine für die *Convergenz* der Reihe $\Sigma(-1)^{\nu}c_{\nu}$ jedenfalls *nothwendige* ist, so gilt dies *a fortiori* von der Beziehung:

$$\mathop{\mathrm{L}}_{n=\infty}b_{n}\sum_{\kappa=0}^{n}a_{\kappa}=\mathop{\mathrm{L}}_{n=\infty}a_{n}\sum_{\kappa=0}^{n}b_{\kappa}=0.$$

MÜNCHEN, 31. März 1901.

UEBER DEN GOURSAT'SCHEN BEWEIS DES CAUCHY'SCHEN INTEGRALSATZES*

VON

ALFRED PRINGSHEIM

Der CAUCHY'sche Satz über das Verschwinden eines geschlossenen Integrals von der Form $\int f(z)\,dz$ hat durch die in Bd. 1, pp. 14–16 dieser Zeitschrift von Herrn GOURSAT publicirte Note eine ausserordentlich bemerkenswerthe Erweiterung erfahren, insofern alle bisherigen Beweise, um vollständig und exakt zu sein, die *Stetigkeit* von $f'(z)$ oder, was im wesentlichen auf dasselbe hinausläuft,† die *gleichmässige* Differenzirbarkeit von $f(z)$, entweder schlechthin oder zum mindesten in gewissem Umfange ‡ zur unentbehrlichen Voraussetzung hatten. In jener Note zeigt nun Herr GOURSAT, dass man denjenigen Hauptschluss, welcher bei dem früher (Acta math., T. IV (1884), pp. 197–200) von ihm mitgetheilten Beweise auf der Voraussetzung der *gleichmässigen* Differenzerbarkeit beruht, mit Hülfe einer weniger speciellen, *jeder* (im complexen Sinne) *differenzirbaren* Function *eo ipso* zukommenden Eigenschaft herleiten kann. Dieselbe ist enthalten in dem folgenden LEMMA:

Es sei $f(z)$ für alle Stellen eines von der geschlossenen Curve C begrenzten Bereiches A stetig und mit einer endlichen Derivirten $f'(z)$ begabt. Wird dann $\epsilon > 0$ beliebig vorgeschrieben. so lässt sich A (auf unendlich viele Arten) in hinlänglich kleine Theilstücke A_ν zerlegen, derart dass:

$$(\mathrm{I}) \qquad |f(z_\nu) - f(\xi_\nu) - (z_\nu - \xi_\nu) f'(\xi_\nu)| \leqq \epsilon\,|z_\nu - \xi_\nu|.$$

Dabei bedeutet z_ν jeden beliebigen Punkt auf der Begrenzung von A_ν, ξ_ν einen bestimmten, im Innern oder auf der Begrenzung von A_ν allemal wirklich vorhandenen Punkt.

Nachdem Herr GOURSAT dieses Lemma bewiesen, begnügt er sich im übrigen mit der Bemerkung, man könne nun auf Grund desselben die für seinen oben citirten, ersten Beweis erforderliche Theilung des Gesammtbereiches immer so

* Presented to the Society at the Ithaca meeting, August 19, 1901. Received for publication May 24, 1901.

† Vgl. Münchener Berichte, Bd. 25 (1895), p. 295.

‡ Vgl. Münchener Berichte, Bd. 29 (1899), p. 61. W. F. OSGOOD, American M. S. Bulletin, Bd. 5 (1898), p. 85.

einrichten, *dass das übrige Raisonnement in nichts geändert zu werden brauche.** Dieser *cum grano salis* zwar zutreffende Ausspruch ist indessen seinem *Wortlaute* nach nicht ganz exakt, ja sogar dazu angethan, † bezüglich der *Tragweite* des obigen Lemmas und der *richtigen Art seiner Anwendung* gewisse Missverständnisse und Zweifel hervorzurufen, ‡ nämlich: Der ursprüngliche GOURSAT'sche Beweis des CAUCHY'schen Satzes beruht auf einer Theilung des Bereiches A in lauter *congruente*, einen gewissen Kleinheitsgrad § besitzende *Quadrate* bezw. (längs des Randes) *Bruchstücke solcher Quadrate*, wobei aber jedes in Betracht kommende Quadrat immer nur *ein* solches Bruchstück liefert, und nicht in *der* Weise zerfallen soll, dass *mehrere* Bruchstücke dem Bereiche A angehören. (Ich will diese Art der Theilung im folgenden stets durch den Ausdruck bezeichnen: es werde der Bereich A in Quadrate "*etc.*" zerlegt).

Fasst man also den obigen Ausspruch des Herrn GOURSAT nach seinem *Wortlaute* auf, so müsste man annehmen, es existire auf Grund jenes Lemmas allemal eine Eintheilung in lauter *congruente* Quadrate "*etc.*" A_ν, derart dass für jedes A_ν die Bedingung (I) erfüllt ist. Dies geht nun aber aus dem *Beweise* des fraglichen Lemmas *keineswegs* hervor, vielmehr lehrt derselbe nur so viel, dass *unter einer gewissen der Randcurve C aufzuerlegenden Beschränkung* zwar stets *quadratische* Eintheilungen der verlangten Art vorhanden sind, dass dieselben aber im allgemeinen aus Quadraten bezw. Bruchstücken von Quadraten *verschiedener* Grössen bestehen werden.

Vor allem hat man die Randcurve C als *abtheilungsweise monoton* d. h. *so* anzunehmen, dass jede ihrer beiden Coordinaten nur eine endliche Anzahl von Extremen aufzuweisen hat. ‖ Man kann alsdann eine Strecke λ_0 hinlänglich

* "sans modifier en rien le reste du raisonnement." (a. a. O., p. 16.)

† It is understood that the criticism relates merely to verbal form, and that neither the scope of GOURSAT'S lemma nor its manner of application in proof of GOURSAT'S form of CAUCHY'S theorem is subject to reasonable doubt. The lemma neither in statement nor in proof refers to an ultimate division into *congruent* squares, while the division into such squares utilized in GOURSAT'S original proof of CAUCHY'S theorem is evidently not an essential part of that proof. In view however of the importance of the subject the application of the lemma may well be given explicitly; cf. p. 416 of the text.—THE EDITORS.

‡ Herr E. V. HUNTINGTON (Strassburg) hatte die Güte, mich hierauf aufmerksam zu machen.

§ Ein von vornherein bestimmter (nämlich von der Natur der Randcurve abhängiger) Kleinheitsgrad des quadratischen Gitters ist erforderlich, wenn jedes am *Rande* auftretende Theilstück lediglich von *einem* zusammenhängenden Stücke der Randcurve, im übrigen von einem *Theile* eines Gitterquadrates begrenzt sein soll. Man kann im übrigen diese (von GOURSAT *stillschweigend*, von O. STOLZ, *Grundzüge der Diff.- und Integr.-R.*, II, pp. 213, 218, *ausdrücklich* gemachte) Voraussetzung in gewissem Umfange auch fallen lassen, ohne das betreffende Raisonnement im wesentlichen zu ändern. (Siehe weiter unten).

‖ Es würde nicht einmal genügen anzunehmen: Die Curve C soll von jeder zu den Axen parallelen Graden nur in einer *endlichen* Anzahl von Punkten *geschnitten* werden (wobei eine Gerade welche die Curve in unendlich vielen Punkten *berührt* oder durch unendlich viele *Spitzen* geht nicht als *Schnitt*linie anzusehen wäre): dabei braucht nämlich die Anzahl der Schnittpunkte

klein so fixiren, dass kein Quadrat, dessen Seiten $\leqq \lambda_0$ und den Coordinatenaxen parallel sind, von der Randcurve in mehr als zwei Stücke zerschnitten wird. Denkt man sich nun den Bereich A zunächst mit einem quadratischen Gitter von der Seitenlänge $\lambda \leqq \lambda_0$ überzogen, so folgt aus der beim Beweise des GOURSAT'schen Lemmas angewendeten Schlussweise für's erste nur so viel, dass bei hinlänglicher Verkleinerung von λ (etwa durch Untertheilung von λ_0) *ein* oder *mehrere* Theilbereiche A_ν der Bedingung (I) genügen müssen; sodann, dass jeder Theilbereich A'_ν, welcher der Bedingung (I) noch nicht genügt, durch weitere geeignete Untertheilung im Quadrate "*etc.*" mit der Eigenschaft (I) zerlegt werden könne. Es folgt aber *nicht*, dass die für irgend ein A'_ν geeignete Untertheilung dieses A'_ν in lauter *congruente* Quadrate "*etc.*" zerlegen müsse und, selbst wenn dies der Fall sein sollte, dass die Anwendung der *nämlichen* Untertheilung auf irgend einen der mit A_ν bezeichneten, *schon* der Bedingung (I) *genügenden* Theilbereiche *wieder* lauter Theilbereiche *mit der Eigenschaft* (I) liefern müsse.

Im übrigen wird man jene "geeignete" Untertheilung am einfachsten in *der* Weise erzielen, dass man jedes der Quadrate, welches einen Theilbereich A'_ν ausmacht oder in sich enthält, in 4 congruente Quadrate mit der Seitenlänge $\lambda/2$ zerlegt, jedes dieser letzteren, *sofern es nicht bereits einen der Bedingung* (I) *genügenden Theilbereich liefert* oder überhaupt keine Innenpunkte von A mehr enthält, wieder in 4 congruente Quadrate mit der Seitenlänge $\lambda/4$ u. s. f. Der GOURSAT'sche Schluss lehrt dann, dass der angedeutete Process, bei dem also jedesmal alle diejenigen Quadrate, welche der Bedingung (I) bereits genügende Theilbereiche liefern oder keine Innerpunkte von A mehr enthalten, *von der weiteren Untertheilung ausgeschlossen sind*, nach einer *begrenzten* Anzahl von Operationen zu dem gewünschten Endziele führen muss. Man gelangt also auf diese Weise zu der folgenden pseudelleren Formulirung des fraglichen Lemmas.

Unter den oben über $f(z)$ und über die Randcurve C gemachten Voraussetzungen lässt sich der Bereich A in eine endliche Anzahl von Quadraten Q_μ und (oben näher qualificirten) Bruchstücken R_ν von Quadraten mit der Sei-

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noch *nicht* (wie bei einer *abtheilungsweise monotonen* Curve) unter einer *festen* endlichen Zahl zu bleiben (*Beispiel:* $y = x^2 \sin^2 1/x$ in jedem endlichen die Stelle $x = 0$ enthaltenden Intervall). In Folge dessen könnten sich bei der Untertheilung die Schnittpunkte jedesmal so vermehren, dass am Rande von A immer noch Quadrate vorhanden bleiben, welche *mehrere* Theilstücke von A enthalten. Man könnte indessen in diesem Falle dem fraglichen Lemma die folgende, für den Beweis des Integralsatzes noch ausreichende (s. die folgende Fussnote) Form geben: „*Der Bereich A lässt sich mit einer endlichen Anzahl von Quadraten Q_ν überdecken, derart dass jedes Q_ν nur eine endliche Anzahl von Theilstücken A_{m_ν} des Bereiches A und überdies einen ebenfalls zu A (also zu einem der A_{m_ν}) gehörigen Punkt ξ_ν von folgender Eigenschaft enthält: man hat*

$$|f(z_{m_\nu}) - f(\xi_\nu) - (z_{m_\nu} - \xi_\nu) \cdot f'(\xi_\nu)| \leqq \varepsilon |z_{m_\nu} - \xi_\nu|,$$

wobei wiederum z_{m_ν} *jeden Punkt der Begrenzung aller A_{m_ν} bedeutet.*"

tenlänge $\lambda, \lambda/2, \cdots, \lambda/2^m$ *zerlegen, derart dass jedes* Q_μ *und* R_ν *der Bedingung* (I) *genügt.*

Legt man nun bei dem GOURSAT'schen Beweise des Integralsatzes eine in vorstehender Weise *modificirte* Gebietstheilung zu Grunde, so behalten nichtsdestoweniger alle *wesentlichen* Schlüsse ihre Geltung, da sie in Wahrheit von einer etwaigen *Congruenz* der betreffenden Quadrate völlig unabhängig sind. Man hat nämlich für jedes der oben mit Q_μ bezeichneten quadratischen Theilstücke, falls dessen Seitenlänge mit a_μ bezeichnet wird:

$$\int_{(Q_\mu)} |z_\mu - \xi_\mu| \, |dz_\mu| < (\sqrt{2}\, a_\mu) 4 a_\mu = \sqrt{2}\, 4\, Q_\mu .$$

Und wenn R_ν ein Bruchstück eines Quadrates $Q_\nu = a_\nu^2$ bedeutet, welches zum Theil von dem Curvenstücke s_ν begrenzt wird:

$$\int_{(R_\nu)} |z_\nu - \xi_\nu| \, |dz_\nu| < (\sqrt{2}\, a_\nu)(4 a_\nu + s_\nu) \leqq \sqrt{2}\, (4 Q_\nu + \lambda s_\nu).^*$$

Bezeichnet man also mit $\mathfrak{A}$ die Summe der Quadrate des ursprünglichen λ-Gitters, soweit dieselben Innen-Punkte von A enthalten, mit S die Gesammtlänge der Randcurve von A, und fasst ausserdem wieder die Q_μ, R_ν unter der Beziehung A_ν zusammen, so folgt:

$$\Sigma \int_{(A_\nu)} |z_\nu - \xi_\nu| \, dz_\nu < \sqrt{2}\, (4 \mathfrak{A} + \lambda S),$$

worauf dann alles übrige genau wie bei dem ursprünglichen GOURSAT'schen Beweise sich ergiebt.

Ich glaube nicht fehl zu gehen, wenn ich annehme, dass Herr GOURSAT den oben citirten Ausspruch ungefähr im Sinne der eben gemachten Auseinandersetzungen verstanden wissen will: immerhin dürften die vorstehenden Erläuterungen wohl nicht so ganz selbstverständlich und somit auch nicht überflüssig erscheinen.

Der Vollständigkeit halber möchte ich noch hinzufügen, dass die für den obigen Beweis, noch erforderliche Relation:

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* Dabei könnte R_ν ausser der von Herrn GOURSAT (Acta Mathematica, a. a. O., p. 199) schematisch angedeuteten Gestalt (Fig. I) auch diejenige von Fig. II haben. Die obige Ungleichung bleibt aber auch richtig für ein Theilstück von der Form Fig. III und sogar, wenn der in der vorigen Fussnote charakterisirte Fall vertritt (Fig. IV): natürlich hat man dann unter s_ν die *Summe* der betreffenden Curvenbögen zu verstehen.

$$\int_{(c)} z\,dz = 0$$

(wo c eine geschlossene Curve bedeutet, die man im übrigen nur als stetig und rectificirbar anzunehmen braucht) entgegen einer früher von mir gemachten, die principielle Einfachheit des GOURSAT'schen Beweises mit Unrecht in Frage stellenden Bemerkung,* *ohne* jede umständlichere Grenzbetrachtung auf die allereinfachste Art gewonnen werden kann, sofern nur die Bedeutung und eindeutige Existenz von $\int f(z)\,dz$, erstreckt über eine Curve vom Charakter c in der üblichen Weise festgestellt ist. Darnach hat man nämlich, wenn $z_0, z_1, \cdots, z_{n-1}$, $z_n = z_0$ eine cyklische Folge von Punkten auf der Curve c bedeuten und ξ_ν einen willkürlichen Punkt des Bogenstückes $z_{\nu-1}z_\nu$ vorstellt:

$$\int_{(c)} z\,dz = \mathop{\mathrm{L}}_{n=\infty} \sum_{\nu=1}^{n} (z_\nu - z_{\nu-1})\,\xi_\nu \qquad (z_n = z_0).$$

Setzt man jetzt einmal: $\xi_\nu = z_\nu$, darauf: $\xi_\nu = z_{\nu-1}$, so folgt durch Addition der betreffenden Gleichungen:

$$2\int_{(c)} z\,dz = \mathop{\mathrm{L}}_{n=\infty} \sum_{\nu=1}^{n} (z_\nu^2 - z_{\nu-1}^2) = \mathop{\mathrm{L}}_{n=\infty} (z_n^2 - z_0^2) = 0, \quad \text{q.e.d.}$$

Durch die vorstehenden Betrachtungen wird der fragliche Integralsatz zunächst nur für den Fall *abtheilungsweise monotoner* Randcurven † gewonnen. Will man *rectificirbare* Curven im *allgemeinsten* (d. h. SCHEEFFER-JORDAN'schen) Sinne als Begrenzung zulassen ‡ so bleibt wohl kaum ein anderer Weg, als dass man dem obigen Resultate lediglich die Gültigkeit des Satzes für ein *Dreieck* entnimmt und sodann mit Hülfe einer Grenzbetrachtung auf *rectificir-*

* Münchener Berichte, Bd. 25 (1895), p. 42, Fussnote.

† Bezw. für Randcurven, welche von keiner Parallelen zu den Axen in unendlich vielen Punkten geschnitten werden.

‡ In Bd. 1 dieser Zeitschrift, p. 501, letzte Fussnote, bemerkt Herr E. H. MOORE, ich scheine leider mit dem heutzutage üblichen Begriffe der *allgemeinen rectificirbaren Curven* nicht in Fühlung zu sein ("*out of touch with* * * *"). Das dürfte indessen wohl nicht ganz zutreffen. Ich glaube sogar in Folge des intimen Umganges mit dem Schöpfer jenes Begriffes, meinem leider allzu früh verstorbenen Freunde LUDWIG SCHEEFFER, hierüber ebenso früh und ebenso genau orientirt gewesen zu sein, als irgend ein anderer. Immerhin hatte ich, als ich den von Herrn MOORE citirten Aufsatz (Münchener Berichte, 1895) über den CAUCHY'schen Integralsatz schrieb, nach dem Vorgange von DU BOIS-REYMOND gegen die definitive Einführung des verallgemeinerten Rectifications-Begriffes gewisse Bedenken (a. a. o., p. 55, Fussnote 2). Seitdem ist nun freilich mein Glaube an des letzteren Autorität, wie in manchen anderen Punkten, so auch in *diesem* merklich geschwunden. Und nachdem durch die *zweite* Auflage von C. JORDAN'S *Cours d'Analyse* T. I (welche zur Zeit der Abfassung jenes Aufsatzes zwar schon erschienen, mir jedoch unbekannt geblieben war) die *Fruchtbarkeit* des fraglichen Begriffes in das richtige Licht gesetzt worden ist, habe ich denselben auch bedingungslos acceptirt (s. z. B. Encyklop. der Math. W., Bd. II, p. 41).

bare Curven überträgt. * Hat man also dieses Ziel vor Augen, so erscheint es offenbar weit zweckmässiger, jenen Integralsatz zunächst *überhaupt nur* für den wesentlich einfacher gearteten Fall eines *Dreiecks* zu beweisen. Nach meinem Dafürhalten würde aber dieser Weg selbst dann jedem anderen vorzuziehen sein, wenn man sich—etwa aus didaktischen Rücksichten—schliesslich mit der Zulassung *abtheilungsweise monotoner* oder auch nur "*gewöhnlicher*" Randcurven begnügen will. Die Definition bezw. der Existenz-Beweis für das über eine Curve erstreckte Integral $\int f(z)\,dz$ erfordert lediglich die *Stetigkeit*, nicht aber die *Differenzirbarkeit* von $f(z)$. Auf *derselben* Grundlage ergiebt sich, dass eine infinitesimale Aenderung des Integrationsweges auch nur eine infinitesimale Aenderung des Integralwerthes nach sich zieht, dass also in's besondere jedes *Curven*-Integral mit beliebiger Annäherung durch ein *Polygon*-Integral ersetzt werden kann. Diese Erkenntniss ist *an sich* wichtig und für zahlreiche analytische Untersuchungen geradezu *unentbehrlich*. Hat man sie nun aber einmal gewonnen, so liegt in der That gar kein vernünftiger Grund vor, den Beweis des eigentlichen Integralsatzes mit denjenigen Complicationen zu belasten, die sich aus der mehr oder minder zusammengesetzten Natur der Randcurve ergeben. Der wahre *Kern* jenes Integralsatzes liegt in seiner Gültigkeit für irgend einen *Special*-Bereich *einfachster* Art z. B. ein *Dreieck* und unter Voraussetzung der *Differenzirbarkeit* von $f(z)$. Die Möglichkeit, ihn auf *krummlinig* begrenzte Bereiche zu übertragen, beruht dagegen lediglich auf *Stetigkeits*-Eigenschaften, welche den Integralen *jeder stetigen* Function zukommen. †

Dies vorausgeschickt, glaube ich vielleicht manchem einen kleinen Dienst zu leisten, wenn ich den fraglichen „*Dreiecks*"-Beweis, einschliesslich der hierfür zweckmässigen Formulirung und Herleitung des GOURSAT'schen Lemmas im folgenden noch vollständig durchführe.

LEMMA I. (*Specialform des Goursat'schen Lemmas*). *Ist $f(z)$ für jede einzelne Stelle z' im Innern und auf der Begrenzung des Dreiecks Δ eindeutig definirt, stetig und mit einem endlichen ‡ Differentialquotienten $f'(z)$ begabt, so*

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* Vgl. C. JORDAN, a. a. O., § 196, 197. Auch der in Bd. 1 dieser Zeitschrift, pp. 499–506 mitgetheilte Beweis des Herrn MOORE liefert in der bezeichneten Richtung keinen anderen Weg (vgl. a. a. o., p. 505), da hierbei die Randcurven ebenfalls noch von gewissen Special-Bedingung, übrigens von ziemlich complicirtem Charakter genügen müssen.

† In der That braucht, bei der Beschränkung auf *abtheilungsweise monotone* Randcurven C, für die Gültigkeit des Integralsatzes nur die *Stetigkeit*, *nicht* die *Differenzirbarkeit* von $f(z)$ auf C selbst vorausgesetzt zu werden. Vgl. Münchener Berichte, a. a. O., p. 71. (Für den Fall, dass C „Schnabelspitzen" besitzt, wäre noch eine einfache Modification zu dem a. a. o. gesagten hinzuzufügen).

‡ Damit ist also nicht einmal vorausgesetzt, dass die Gesammtheit der Zahlen $|f'(z)|$ *unter einer endlichen Schranke* bleibt. Auch brauchen in Bezug auf die Stetigkeit von $f(z)$ und bei der Bildung von $f'(z)$ durchaus nur solche Punkte in Betracht gezogen zu werden, welche dem Innern oder der Begrenzung von Δ angehören (mit anderen Worten: $f(z)$ braucht „*nach aussen*" weder stetig, noch differenzirbar zu sein).

lässt sich Δ zu jedem $\epsilon > 0$ in eine endliche Anzahl unter sich und mit Δ ähnlicher (im allgemeinen aber nicht congruenter) Dreiecke Δ_ν zerlegen, derart dass:

$$(\mathrm{I}) \qquad |f(z_\nu) - f(\xi_\nu) - (z_\nu - \xi_\nu) f'(\xi_\nu)| \leqq \epsilon |z_\nu - \xi_\nu| .$$

Dabei bedeutet z_ν jeden beliebigen Punkt auf der Begrenzung von Δ_ν, dagegen ξ_ν einen bestimmten, im Inneren oder auf der Begrenzung von Δ_ν allemal wirklich vorhandenen Punkt.

Beweis. Der auf die Existenz von $f'(z)$ bezügliche Theil der Voraussetzung besagt folgendes: Zu jeder Stelle z' und zu jedem $\epsilon > 0$ existirt eine bestimmte Zahl $f'(z')$ und eine positive Zahl ρ'_ϵ, sodass:

$$\left| \frac{f(z) - f(z')}{z - z'} - f'(z') \right| < \epsilon \quad \text{für:} \quad 0 < |z - z'| < \rho'_\epsilon ,$$

und daher:

$$(1) \qquad |f(z) - f(z') - (z - z') f'(z')| \leqq \epsilon |z - z'| \quad \text{für:} \quad 0 \leqq |z - z'| < \rho'_\epsilon .$$

Man halbire nun zunächst die 3 Seiten $a^{(1)}$, $a^{(2)}$, $a^{(3)}$ von Δ und zerlege Δ durch geradlinige Verbindung der Halbirungs-Punkte in 4 congruente, dem ursprünglichen Δ ähnliche Dreiecke mit den Seiten $a^{(\kappa)}/2$ $(\kappa = 1, 2, 3)$. Jedes dieser 4 Dreiecke, sofern es nicht bereits der Bedingung (I) genügt, zerlege man auf die nämliche Art und setze dieses Verfahren analog d. h. in *der* Weise fort, dass jedes zum Vorschein kommende Theildreieck, welches der Bedingung (I) genügt, von der weiteren Untertheilung ausgeschlossen wird. Man muss dann allemal nach einer begrenzten Anzahl solcher Operationen zu einer Eintheilung gelangen, bei welcher *jedes* Theildreieck der Bedingung (I) genügt.

Denn *angenommen*, dies wäre *nicht* der Fall, so müsste wenigstens *eine unbegrenzte* Folge von Dreiecken

$$\Delta, \Delta'_1, \Delta'_2, \cdots, \Delta'_\nu, \cdots$$

mit den resp. Seitenlängen:

$$a^{(\kappa)}, \frac{a^{(\kappa)}}{2}, \frac{a^{(\kappa)}}{2^2}, \cdots, \frac{a^{(\kappa)}}{2^\nu}, \cdots \qquad (\kappa = 1, 2, 3)$$

resultiren, derart dass jedes dieser Dreiecke Δ'_ν einen Bestandtheil (nämlich ein Viertel) des vorangehenden bildet, also auch in *sämmtlichen* vorangehenden enthalten ist, während andererseits *keins* dieser Dreiecke der Bedingung (I) genügt. Es müsste dann *ein* bestimmter Punkt ξ existiren, welcher im Innern oder auf der Begrenzung *sämmtlicher* Δ'_ν (incl. Δ) liegt. Alsdann hätte man aber nach Ungl. (1):

$$(2) \qquad |f(z) - f(\xi) - (z - \xi) f'(\xi)| \leqq \epsilon |z - \xi|$$

für ein gewisses $\rho_\epsilon > 0$ und alle dem Bereiche Δ angehörigen, der Bedingung: $|z - \xi| < \rho_\epsilon$ genügenden Punkte z. Wird nun aber n so gross angenommen, dass: $a^{(\kappa)}/2^n < \rho_\epsilon$ $(\kappa = 1, 2, 3)$, so fallen sämmtliche Δ'_ν, für welche $\nu \geqq n$, *ganz innerhalb* des Geltungsbereiches der Ungleichung (2), würden also speciell auch in Bezug auf ihre *Begrenzungspunkte* der Bedingung (I) genügen, was der oben gemachten Annahme *widerspricht*. Dieselbe erscheint daher als unzulässig, womit dann das ausgesprochene Lemma bewiesen ist.

LEMMA II. *Zerfällt bei der in dem vorigen Lemma beschriebenen Theilung das Dreieck Δ mit dem Umfange $a^{(1)} + a^{(2)} + a^{(3)} = s$ im ganzen in p ähnliche Dreiecke Δ_ν mit den resp. Umfängen s_ν, so ist:*

$$\text{(II)} \qquad \sum_{\nu=1}^{p} s_\nu^2 = s^2,$$

Beweis. Es besitze die *kleinste* Art von Dreiecken, die bei der fraglichen Eintheilung vorkommt, die Seitenlängen $a^{(\kappa)}/2^n$ $(\kappa = 1, 2, 3)$, also den Umfang $s/2^n$. Jedes etwa vorkommende *grössere* Dreiecke Δ_ν besitzt dann einen Umfang: $s_\nu = 2^{m_\nu} \cdot s/2^n$, wo: $1 \leqq m_\nu < n$, und lässt sich darnach in 2^{2m_ν} jener *Minimal*-Dreiecke zerlegen. Da nun: $s_\nu^2 = 2^{2m_\nu} \cdot (s/2^n)^2$, so darf man in dem Ausdrucke: $\sum_{\nu=1}^{\nu=p} s_\nu^2$ jedes s_ν^2, welches nicht schon von einem *Minimal*-Dreiecke Δ_ν herrührt, also den Werth: $1 \cdot (s/2^n)^2$ besitzt, gerade durch dasjenige *Multiplum* von $(s/2^n)^2$ ersetzen, welches die genaue *Anzahl* der in Δ_ν Platz findenden *Minimal*-Dreiecke angiebt. Da aber die Gesammtzahl der in Δ überhaupt unterzubringenden *Minimal*-Dreiecke 2^{2n} ist, so ergiebt sich:

$$\sum_{\nu=1}^{p} s_\nu^2 = 2^{2n} \cdot \left(\frac{s}{2^n}\right)^2 = s^2, \quad \text{q.e.d.}^*$$

HAUPTSATZ. *Ist $f(z)$ für jede einzelne Stelle im Innern und auf der Begrenzung des Dreiecks Δ eindeutig definirt, stetig und mit einem endlichen $f'(z)$ begabt, so ist:*

$$\int_{(\Delta)} f'(z)\, dz = 0.$$

Beweis. Man nehme $\epsilon > 0$ beliebig klein an und denke sich das Dreieck Δ auf Grund des Lemma I in p ähnliche Dreiecke Δ_ν zerlegt, welche der Bedingung (I) genügen. Man hat dann zunächst:

* Der Beweis gestaltet sich noch kürzer, wenn man den *geometrischen* Satz benützen will, dass die Flächen ähnlicher Dreiecke sich verhalten, wei die Quadrate homologer Seiten, also auch wie diejenigen der Perimeter. Mann findet alsdann ohne weiteres:

$$\sum_{\nu=1}^{p} s_\nu^2 : s^2 = \sum_{\nu=1}^{p} \Delta_\nu : \Delta, \quad \text{d.h.} \quad \sum_{\nu=1}^{p} s_\nu^2 = s^2.$$

$$\int_{(\Delta)} f(z)\,dz = \sum_{\nu=1}^{p} \int_{(\Delta_\nu)} f(z)\,dz = \sum_{\nu=1}^{p} \int f(z_\nu)\,dz_\nu ,$$

vorausgesetzt, dass alle Integrale in *demselben*, etwa dem *positiven* Sinne verstanden werden.

Nun ist identisch:

$$\int f(z_\nu)\,dz_\nu = \int \{f(z_\nu) - f(\xi_\nu) - (z_\nu - \xi_\nu) f'(\xi_\nu)\}\,dz_\nu$$

$$+ \{f(\xi_\nu) - \xi_\nu f'(\xi_\nu)\} \int dz_\nu + f'(\xi_\nu) \int z_\nu\,dz_\nu ,$$

und daher, wegen:

$$\int dz_\nu = 0 , \quad \int z_\nu\,dz_\nu = 0 ,$$

und mit Berücksichtigung von Ungl. (I):

$$\left| \int f(z_\nu)\,dz_\nu \right| \leqq \int |f(z_\nu) - f(\xi_\nu) - (z - \xi_\nu) f'(\xi_\nu)|\,|dz_\nu| ,$$

$$\leqq \epsilon \int |z_\nu - \xi_\nu|\,|dz_\nu| .$$

Wird nun wiederum der Umfang von Δ_ν mit s_ν bezeichnet, so hat man unter allen Umständen:

$$|z_\nu - \xi_\nu| < \tfrac{1}{2} s_\nu ,$$

also:

$$\left| \int f(z_\nu)\,dz_\nu \right| < \tfrac{1}{2} \epsilon s_\nu \int |dz_\nu| = \frac{\epsilon}{2} s_\nu^2 ,$$

und daher mit Berücksichtigung von Lemma II:

$$\left| \int_{(\Delta)} f(z)\,dz \right| < \frac{\epsilon}{2} \sum_{\nu=1}^{p} s_\nu^2 = \frac{\epsilon}{2} s^2 ,$$

d. h. schliesslich:

$$\int_{(\Delta)} f(z)\,dz = 0 , \quad \text{q.e.d.}^*$$

MÜNCHEN, 5. Mai, 1901.

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* In ganz analoger Weise kann der Beweis auch für den Fall eines *Rechtecks* geführt werden: der von mir in Bd. I (1900) der Bibliotheca mathematica, p. 478 skizzirte Beweis ist in diesem Sinne zu vervollständigen. In seiner jetzigen Fassung beruht er auf derjenigen unzutreffenden Anwendung des GOURSAT'schen Lemmas, welche auf den ersten Seiten der vorliegenden Note des näheren erörtert wurde.

NEW PROOF OF A THEOREM OF OSGOOD'S IN THE CALCULUS OF VARIATIONS*

BY

OSKAR BOLZA

In the last number of the Transactions,† Professor OSGOOD has proved an important characteristic property of a strong minimum of an integral of the form

$$I = \int_{\tau_0}^{\tau_1} F(x, y, x', y')\, d\tau. \tag{1}$$

His proof, however, is rather complicated, and the following note is intended to give a simpler proof of the theorem.

§ 1.

Introduction of curvilinear coördinates.

Suppose the integral (1) is taken along a continuous curve C with continuously turning tangents:

$$C: \quad x = \phi(\tau), \quad y = \psi(\tau) \qquad (\tau_0 \leqq \tau \leqq \tau_1)$$

joining two fixed points $A(\tau_0)$ and $B(\tau_1)$; further $\phi'^2 + \psi'^2 \neq 0$ in (τ_0, τ_1).

Concerning the function $F(x, y, x', y')$ we make the same assumptions as OSGOOD on p. 277, l. c., except the assumption $F > 0$, which is not necessary for the present proof.

Now introduce instead of the rectangular coördinates x, y any curvilinear coordinates

$$u = U(x, y), \quad v = V(x, y), \tag{2}$$

where U, V are single-valued functions with continuous first and second derivatives in a region T of the x,y-plane containing the curve C; in the same region their Jacobian is supposed $\neq 0$. Interpret u, v as the rectangular co-

* Presented to the Society at the Ithaca meeting, August 19, 1901. Received for publication September 2, 1901.

† Transactions of the American Mathematical Society, vol. 2, p. 273.

ordinates of a point in a u,v-plane and denote by T', C', A', B' the images of T, C, A, B respectively. Suppose further that the correspondence between T and T' is a one-to-one correspondence and that accordingly the inverse functions

$$x = X(u, v), \quad y = Y(u, v), \tag{3}$$

are single-valued functions with continuous first and second derivatives in T' and

$$D = \frac{\partial(X, Y)}{\partial(u, v)} \neq 0$$

in T'.

Then the integral I is changed into

$$I' = \int_{\tau_0}^{\tau_1} G(u, v, u', v')\, d\tau, \tag{4}$$

the function G of the four arguments u, v, u', v' being defined by

$$G(u, v, u', v') \equiv F(X, Y, X_u u' + X_v v', Y_u u' + Y_v v'), \tag{5}$$

where $X_u = \partial X/\partial u$, etc.

The integral I' is taken, in the u,v-plane, along the image C' of C.

From $I = I'$ it follows that if the curve C minimize the integral I, its image C' will minimize I', and vice versa; and if C be an extremal for I, C' must be an extremal for I', and vice versa. Further WEIERSTRASS's function F_1 is an invariant for the above transformation, viz., if we denote the corresponding function derived from G by G_1, we obtain easily

$$G_1 = D^2 F_1. \tag{6}$$

Finally WEIERSTRASS's E-function is an absolute invariant, i. e., if we denote the new E-function by E' we have:

$$\begin{cases} E'(u, v; u', v'; \bar{u}', \bar{v}') = E(x, y; x', y'; \bar{x}', \bar{y}') \\ \qquad\qquad \text{where} \\ x' = X_u u' + X_v v', \quad \bar{x}' = X_u \bar{u}' + X_v \bar{v}', \\ y' = Y_u u' + Y_v v', \quad \bar{y}' = Y_u \bar{u}' + Y_v \bar{v}', \end{cases} \tag{7}$$

as follows immediately from (5).

§ 2.

Proof of Osgood's theorem.

Now let

$$x = \phi(t, a), \quad y = \psi(t, a) \tag{8}$$

be a set of extremals for the integral I, satisfying the following conditions:

1. The functions ϕ, ψ are single-valued functions of t, a with continuous first and second derivatives in the region:

$$T_0 - \epsilon \leqq t \leqq T_1 + \epsilon, \quad |a - a_0| \leqq \kappa \qquad (\epsilon > 0).$$

2. The extremal $C_0 : x = \phi(t, a_0)$, $y = \psi(t, a_0)$ has no multiple point for $T - \epsilon \leqq t \leqq T_1 + \epsilon$, and passes through the two given points $A(t_0)$ and $B(t_1)$ where $T_0 < t_0 < t_1 < T_1$.

3. If we denote by $\Delta(t, a)$ the Jacobian

$$\Delta(t, a) = \frac{\partial(\phi, \psi)}{\partial(t, a)},$$

then

$$\Delta(t, a_0) \neq 0 \qquad \text{in} \qquad (T_0 - \epsilon, \ T_1 + \epsilon). \tag{9}$$

4. The inequality

$$F_1\big(\phi(t, a_0), \ \psi(t, a_0), \ \cos\lambda, \ \sin\lambda\big) > 0 \tag{10}$$

holds for every t of the interval $T_0 - \epsilon \leqq t \leqq T_1 + \epsilon$ and for every λ.

Under these circumstances if we denote by R_κ the region:

$$R_\kappa : \qquad T_0 \leqq t \leqq T_1, \quad |a - a_0| \leqq \kappa,$$

and denote by S_κ the image of R_κ in the x,y-plane, then κ can be taken so small that

1. $\Delta(t, a) \neq 0$ in R_κ.
2. $F_1(x, y, \cos\lambda, \sin\lambda) > 0$ for every x, y in S_κ and for every λ.
3. The correspondence between R_κ and S_κ is a *one-to-one correspondence.**

The inverse functions

$$t = t(x, y), \qquad a = a(x, y)$$

will then be single-valued with continuous first and second derivatives in S_κ and their Jacobian will be $\neq 0$ in S_κ.

Then S_κ is a "field" surrounding the arc AB of the extremal C_0, and if

$$C : \qquad x = \bar\phi(\tau), \quad y = \bar\psi(\tau), \qquad (\tau_0 \leqq \tau \leqq \tau_1),$$

be any other curve with continuously turning tangents drawn from A to B in S_κ, for which $\bar\phi'^2 + \bar\psi'^2 \neq 0$, then WEIERSTRASS's theorem holds, according to which

$$\Delta I = I_{\bar C}(AB) - I_C(AB) = \int_{\tau_0}^{\tau_1} E(x, y; \ x', y'; \ \bar x', \bar y')\, d\tau, \tag{11}$$

* See KNESER, *Lehrbuch der Variationsrechnung*, § 14, and OSGOOD, l. c., p. 278. Both proofs have to be supplemented by the following preliminary lemma: If for every κ, however small, there existed points (x, y) in S_κ to which correspond in R_κ at least two distinct points (t', a') and (t'', a''), then there must exist, in R_κ, a point (t_2, a_0) such that in every vicinity of it pairs of distinct points can be found whose images in the x, y-plane coincide.

where x', y' refer to the extremal of the set (8) passing through x, y, and $\bar{x}'$, $\bar{y}'$ to the curve $\bar{C}$. Hence WEIERSTRASS infers that $\Delta I > 0$ by making use of the following theorem † connecting the functions E and F_1:

$$(12)\quad E(x, y;\ \cos\vartheta, \sin\vartheta;\ \cos\bar{\vartheta}, \sin\bar{\vartheta}) = (1-\cos\omega)\,F_1(x, y, \cos\vartheta^*, \sin\vartheta^*),$$

where $\omega = \bar{\vartheta} - \vartheta$; ϑ^* is an angle between ϑ and $\bar{\vartheta}$; and the angles are so measured that $|\omega| \leqq \pi$.

Now let $0 < h < \kappa$ and use S_h in the analogous signification as S_κ; then OSGOOD'S theorem may be stated as follows: *There exists a positive quantity depending upon h, ϵ_h, such that for every curve C joining A and B drawn in the interior of S_κ but not wholly contained in S_h,*

$$\Delta I > \epsilon_h.$$

To prove this theorem it is now only necessary to introduce instead of x, y the curvilinear coördinates

$$(13)\qquad u = t(x, y), \quad v = a(x, y)$$

which satisfy for the regions S_κ and R_κ the conditions of § 1, and to make use of the remarks made there.

Accordingly the extremals for the integral I' are the lines $v = \text{const.}$, and, therefore, in the u,v-plane WEIERSTRASS'S theorem takes the form:

$$\Delta I = \Delta I' = \int_{s_0}^{s_1} E'(u, v;\ \cos 0, \sin 0;\ \cos\omega, \sin\omega)\,ds$$
$$= \int_{s_0}^{s_1} (1-\cos\omega)\,G_1(u, v, \cos\vartheta^*, \sin\vartheta^*)\,ds.$$

These integrals are taken along the image $\bar{C}'$ in the u,v-plane (i. e., t,a-plane) of the curve $\bar{C}$; ω denotes the angle between the positive tangent to $\bar{C}'$ in the point u, v and the positive u-axis; and s is the arc of the curve $\bar{C}'$.

But from (9), (10), and (6) it follows that we can assign a positive quantity m such that

$$G_1(u, v, \cos\lambda, \sin\lambda) \geqq m > 0$$

for every u, v in R_κ and for every λ. Therefore

$$\Delta I \geqq m\int_{s_0}^{s_1} (1-\cos\omega)\,ds,$$

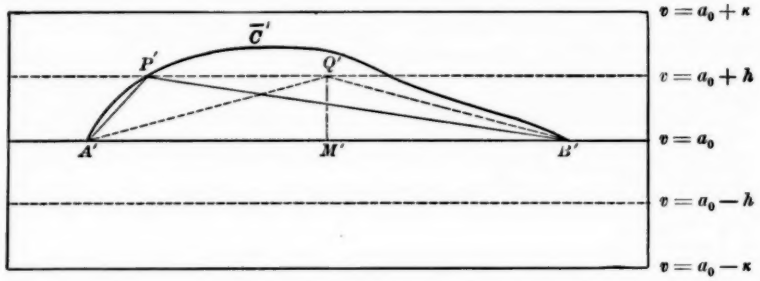
that is

$$\Delta I \geqq m\,(l - (t_1 - t_0)),$$

if l denotes the length of the curve $\bar{C}'$.

† *Lectures on the Calculus of Variations*, 1882.

Now suppose that $\bar{C}$ is not wholly contained in the interior of S_h and therefore passes through a point P of one of the two extremals $a = a_0 \pm h$, then $\bar{C}'$ passes through a point P' whose ordinate is $v = a_0 \pm h$. Hence l is greater



than or equal to the length of the broken line $A'P'B'$. But if we choose Q' on the same line $v = \text{const.}$ as P' so that $A'Q' = B'Q'$, then

$$A'P'B' \geqq A'Q'B',$$

and therefore

$$\Delta I \geqq 2m\left[\sqrt{h^2 + \left(\frac{t_1 - t_0}{2}\right)^2} - \frac{(t_1 - t_0)}{2}\right],$$

which proves OSGOOD's theorem. *

OSGOOD's theorem can easily be extended to the case where one endpoint, say B, is fixed, the other, A, movable on a given curve. For this purpose, it is only necessary to choose for the new coördinates KNESER's curvilinear coördinates u, v (KNESER, § 16); a slight modification of the above reasoning leads then to the inequality

$$\Delta I \geqq m[\sqrt{h^2 + (u_1 - u_0)^2} - (u_1 - u_0)].$$

* OSGOOD bases his proof upon the following lemma: "Let $f(x)$ be a single valued continuous function of x in the interval $a \leqq x \leqq b$, and let $f(x)$ have a continuous derivative $f'(x)$ at all points of this interval. Let $a < l \leqq b$ and $|f(l) - f(a)| = L > 0$. Then

$$\int_a^l f'(x)^2 dx \geqq \frac{L^3}{54(b-a)^2},$$

provided that $L < 3\sqrt{2}(b-a)$."

This lemma may be proved as follows: Since the value of the integral does not change if $f(x)$ is replaced by $M \pm f(x)$, M being a constant, we may confine ourselves to functions $f(x)$ for which $f(a) = 0$, $f(l) = L$. This remark reduces the question to the problem of minimizing the above integral with these given initial values. The solution is $y = L(x-a)/(l-a)$; it satisfies WEIERSTRASS' sufficient conditions for a minimum, and furnishes for the integral the minimum value: $(L^2)/(l-a) \geqq (L^2)/(b-a)$, which under the above inequality for L is greater than $\frac{1}{54}(L^3)/(b-a)^2$.

It is, however, to be remembered that the introduction of KNESER's coördinates presupposes that

$$F\left(\phi(t, a),\ \psi(t, a),\ \phi'(t, a),\ \psi'(t, a)\right) > 0$$

in the region R_κ.

UNIVERSITY OF CHICAGO,
August 31, 1901.

ON CERTAIN PAIRS OF TRANSCENDENTAL FUNCTIONS WHOSE ROOTS SEPARATE EACH OTHER*

BY

MAXIME BÔCHER

Let us consider eight functions $p, q, \phi_1, \phi_2, \psi_1, \psi_2, \chi_1, \chi_2$ of the real variable x, which throughout the interval

$$(I) \qquad a \leqq x \leqq b$$

are real and continuous, and of which the last six have finite first derivatives at every point of this interval. Let y be a solution, which we assume to be not identically zero, of the differential equation

$$(1) \qquad y'' + py' + qy = 0.$$

The functions whose roots we wish to consider in this paper are the following:

$$\Phi = \phi_2 y' - \phi_1 y, \quad \Psi = \psi_2 y' - \psi_1 y, \quad \mathrm{X} = \chi_2 y' - \chi_1 y.$$

It should be noticed that, as a consequence of the assumptions made above, these functions are continuous and have finite first derivatives at every point of the interval (I).

The roots of functions of this form were considered by STURM† by a method which, however, yields results distinctly less general than those we shall obtain. The relation of our methods and results to those of STURM will be indicated at the proper places.

The third section of the present paper may appear at first sight to be of slight importance, the generalizations of the results of the first two sections which are contained in it being in themselves not very far-reaching. The method used in establishing these generalizations is, however, I believe, a valuable one apart from the special application here made of it. I have indicated one application to other questions in the footnote on p. 435.

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† Liouville's Journal, vol. 1 (1836), pp. 149–164.

§ 1. *Two functions.*

We shall consider in the present section only the two functions Φ, Ψ. These functions satisfy a differential equation which may be obtained as follows:

We start from the identity:

$$\Phi(\psi_2 y' - \psi_1 y) - \Psi(\phi_2 y' - \phi_1 y) = 0.$$

From this we obtain by differentiation the equation:

$$(\Phi'\psi_2 - \Psi'\phi_2 + \Phi\psi_2' - \Psi\phi_2' - \Phi\psi_1 + \Psi\phi_1 - p\Phi\psi_2 + p\Psi\phi_2)y' - (\Phi'\psi_1 - \Psi'\phi_1 + \Phi\psi_1' - \Psi\phi_1' + q\Phi\psi_2 - q\Psi\phi_2)y = 0.$$

Since y and y' cannot both vanish at any point of (I) we can eliminate them between the last two equations. This gives:

$$(2)\quad (\phi_1\psi_2 - \phi_2\psi_1)(\Phi'\Psi - \Phi\Psi') + \{\psi_1, \psi_2\}\Phi^2 - \{\psi_1, \phi_2, \phi_1, \psi_2\}\Phi\Psi + \{\phi_1, \phi_2\}\Psi^2 = 0,$$

where for the sake of abbreviation we have set

$$\{\psi_1, \psi_2\} = \psi_1'\psi_2 - \psi_1\psi_2' + \psi_1^2 + p\psi_1\psi_2 + q\psi_2^2,$$

$$\{\psi_1, \phi_2, \phi_1, \psi_2\} = \psi_1'\phi_2 - \psi_1\phi_2' + \phi_1'\psi_2 - \phi_1\psi_2' + 2\phi_1\psi_1 + p(\psi_1\phi_2 + \phi_1\psi_2) + 2q\phi_2\psi_2,$$

$$\{\phi_1, \phi_2\} = \phi_1'\phi_2 - \phi_1\phi_2' + \phi_1^2 + p\phi_1\phi_2 + q\phi_2^2.$$

Equation (2) is the differential equation we were seeking. It may be termed a homogeneous Riccati's equation for the pair of functions Φ, Ψ.*

If the two functions Φ and Ψ were to vanish at the same point we should have at this point

$$\phi_2 y' - \phi_1 y = 0,$$

$$\psi_2 y' - \psi_1 y = 0,$$

and, since y and y' cannot both vanish at this point, as otherwise y would be identically zero, we must have at this point

$$\phi_1\psi_2 - \phi_2\psi_1 = 0.$$

---

* By a Riccati's equation is ordinarily understood an equation of the form:

$$\omega' = A + B\omega + C\omega^2.$$

If for ω we write ω_1/ω_2 this equation takes the homogeneous form:

$$\omega_1'\omega_2 - \omega_1\omega_2' = A\omega_2^2 + B\omega_1\omega_2 + C\omega_1^2,$$

and this is the form of equation (2). Conversely if we divide (2) by Φ^2 and Ψ^2 respectively we obtain at once the ordinary Riccati's equation for Ψ/Φ and Φ/Ψ.

It should be noticed that the function $\{\phi_1, \phi_2\}$ (or $\{\psi_1, \psi_2\}$) when equated to zero gives the homogeneous form of the Riccati's equation which is satified by y'/y.

By assuming, as we shall now do, that $\phi_1\psi_2 - \phi_2\psi_1$ does not vanish at any point of (I) we make it impossible for Φ and Ψ to vanish together.

In this section and the next we shall not use equation (2) itself, but only the following formulæ which follow from it at once:

At a point where $\Phi = 0$:

$$\Phi' = -\frac{\{\phi_1, \phi_2\}}{\phi_1\psi_2 - \phi_2\psi_1}\Psi. \tag{3}$$

At a point where $\Psi = 0$:

$$\Psi' = \frac{\{\psi_1, \psi_2\}}{\phi_1\psi_2 - \phi_2\psi_1}\Phi. \tag{4}$$

Since Φ and Ψ cannot vanish together we see from (3) that, if $\{\phi_1, \phi_2\}$ does not vanish in (I), Φ and Φ' cannot vanish together, and therefore Φ cannot vanish an infinite number of times in (I). Whatever the functions ϕ_1 and ϕ_2 may be, provided merely they do not both vanish at any point of (I), the functions ψ_1 and ψ_2 can be so chosen (e. g., $\psi_1 = -\phi_2$, $\psi_2 = \phi_1$) that $\phi_1\psi_2 - \phi_2\psi_1$ does not vanish in (I). Moreover if ϕ_1 and ϕ_2 vanished at the same point, $\{\phi_1, \phi_2\}$ would also vanish there. We have therefore the theorem which it will be noticed relates to the function Φ alone:

I. *If $\{\phi_1, \phi_2\}$ does not vanish in (I), then Φ cannot vanish more than a finite number of times there and Φ and Φ' do not both vanish at any point of (I).**

Since at two successive roots of Φ the derivative Φ' has opposite signs, and since by (3) this can happen only when Ψ has opposite signs at these two points, we have the theorem:

II. *If neither of the two functions*

$$\phi_1\psi_2 - \phi_2\psi_1, \quad \{\phi_1, \phi_2\}$$

vanishes in (I), then in any portion of (I) in which Ψ does not vanish Φ cannot vanish more than once.†

---

* STURM (l. c., p. 150) proves this theorem for the special case $\phi_2 = e^{\int p dx}$ by using the special case of formula (2) obtained by letting $\psi_2 = 0$, $\psi_1 = -1$. This special case, written to be sure in a slightly modified form, is his equation (11). STURM also restricts himself to the case $\{\phi_1, \phi_2\} > 0$ (see his inequality (14)) and can thus add to our theorem the statement that Φ' and $y = \Psi$ have opposite signs when $\Phi = 0$. He does not point out the connection between this formula and the Riccati's equations which he uses on p. 159.

† This theorem is proved by STURM (l. c., p. 160) by a wholly different method, but only in the very special case

$$\psi_2 = 0, \ \psi_1 = -1, \ \phi_2 = e^{\int p dx}, \ q > 0, \ \phi_1' \geqq 0.$$

This is the more remarkable as his formula (11) would have yielded him the result in the much more general case in which his last two conditions are replaced by the single condition $\{\phi_1, \phi_2\} > 0$. This would have brought his later results (pp. 158–164) into harmony with those which immediately precede them.

A similar result of course holds for the roots of the function Ψ, and by combining these facts we get the theorem:

III. *If none of the functions*

$$\phi_1\psi_2 - \phi_2\psi_1, \quad \{\phi_1, \phi_2\}, \quad \{\psi_1, \psi_2\}$$

vanish in (I) *then between two roots of* Φ *lies one and only one root of* Ψ, *and between two roots of* Ψ *lies one and only one root of* Φ.*

That is, the roots of Φ and Ψ separate each other.

Let us now consider two points at one of which Φ at the other Ψ vanishes while neither function vanishes between them. These points are two successive roots of the product $\Phi\Psi$, and therefore the derivative $\Phi'\Psi + \Phi\Psi'$ has opposite signs at them. From this it follows, as a glance at (3) and (4) shows, that if none of the functions

$$\phi_1\psi_2 - \phi_2\psi_1, \quad \{\phi_1, \phi_2\}, \quad \{\psi_1, \psi_2\}$$

vanish in (I) then $\{\phi_1, \phi_2\}$ and $\{\psi_1, \psi_2\}$ must have the same sign. If, then, these two functions have opposite signs, our assumption that there exist two points, at one of which Φ at the other Ψ vanishes, must be false, and, therefore, by theorem III, neither of these functions can vanish twice. Hence the theorem:

IV. *If none of the functions:*

$$\phi_1\psi_2 - \phi_2\psi_1, \quad \{\phi_1, \phi_2\}, \quad \{\psi_1, \psi_2\}$$

vanish in (I), *and if the last two of them have opposite signs, then neither* Φ *nor* Ψ *vanishes more than once in* (I), *and if one of these functions vanishes the other does not.*

A special case of this theorem deserves mention, namely that in which $\Psi = y$. In order that $\phi_1\psi_2 - \phi_2\psi_1$ shall not vanish, it is necessary and sufficient that ϕ_2 do not vanish in (I). If then we let $\phi = \phi_1/\phi_2$ we have the theorem:

V. *If* ϕ *is continuous, has a finite derivative at every point of* (I), *and satisfies the inequality:*

$$\{\phi, 1\} = \phi' + \phi^2 + p\phi + q < 0,$$

then neither y *nor* $y' - \phi y$ *vanishes more than once in* (I), *and if one of these functions vanishes the other does not.*†

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* This theorem is also proved by STURM (l. c., pp. 161, 163) first in a case analogous to the one mentioned in the last footnote, then in the more general, but still very special, case

$$\phi_2 = \psi_2 = e^{\int p\,dx}, \; q > 0, \; \phi_1' \gtrless 0, \; \psi_1' \gtrless 0.$$

It should be noticed that the condition $\phi_1\psi_2 - \phi_2\psi_1 \neq 0$ now means that ϕ_1 is either everywhere greater or everywhere less than ψ_1.

† The part of this theorem which says that y cannot vanish more than once was proved by the writer by another method in the Bulletin of the American Mathematical Society, May, 1901, p. 335, footnote. Cf. also a less far reaching theorem of PICARD there quoted.

Thus for instance if $q < 0$, y and y' cannot both vanish in (I) nor can either of them vanish more than once.

§ 2. *Three functions.*

We will now establish the following theorem concerning the three functions Φ, Ψ, X.

VI. *If none of the six functions:*

$$\phi_1\psi_2 - \phi_2\psi_1, \quad \psi_1\chi_2 - \psi_2\chi_1, \quad \chi_1\phi_2 - \chi_2\phi_1, \quad \{\phi_1, \phi_2\}, \quad \{\psi_1, \psi_2\}, \quad \{\chi_1, \chi_2\}$$

vanish in (I), *if the last three of them have the same sign, and if the product of all six is negative, then between any root of* Φ *and a larger root of* X *lies a root of* Ψ, *between any root of* Ψ *and a larger root of* Φ *lies a root of* X, *and between any root of* X *and a larger root of* Ψ *lies a root of* Φ.*

Let us write:

$$i = \operatorname{sgn}\{\phi_1, \phi_2\} = \operatorname{sgn}\{\psi_1, \psi_2\} = \operatorname{sgn}\{\chi_1, \chi_2\}.\dagger$$

Then the assumption that the product of the six functions is negative is expressed by the equation:

$$i \cdot \operatorname{sgn}(\phi_1\psi_2 - \phi_2\psi_1) \cdot \operatorname{sgn}(\psi_1\chi_2 - \psi_2\chi_1) \cdot \operatorname{sgn}(\chi_1\phi_2 - \chi_2\phi_1) = -1.$$

Let x_1 and x_2 be two points of (I) such that

$$\Phi(x_1) = 0, \quad X(x_2) = 0, \quad x_1 < x_2.$$

If Φ vanished between x_1 and x_2, say at x', Ψ would vanish, by theorem II, between x_1 and x'. If Φ does not vanish between x_1 and x_2 we may proceed as follows:

Since $\operatorname{sgn}\Phi(x_1 + \epsilon) = \operatorname{sgn}\Phi'(x_1)$ we see by formula (3) that

$$\operatorname{sgn}\Phi(x_1 + \epsilon) = -i \cdot \operatorname{sgn}(\phi_1\psi_2 - \phi_2\psi_1) \cdot \operatorname{sgn}\Psi(x_1). \tag{5}$$

Furthermore if in the identical relation

$$(\phi_1\psi_2 - \phi_2\psi_1)X + (\chi_1\phi_2 - \chi_2\phi_1)\Psi + (\psi_1\chi_2 - \psi_2\chi_1)\Phi = 0 \tag{6}$$

we let $x = x_2$ so that the first term drops out, and then note the signs of the two remaining terms, we get:

$$\operatorname{sgn}\Psi(x_2) = -\operatorname{sgn}(\psi_1\chi_2 - \psi_2\chi_1) \cdot \operatorname{sgn}(\chi_1\phi_2 - \chi_2\phi_1) \cdot \operatorname{sgn}\Phi(x_2). \tag{7}$$

* A special case of this theorem and the next, analogous to the ones mentioned in previous footnotes, is proved by STURM (l. c., p. 165).

† We understand with KRONECKER by sgn Ω the value $+1$ or -1 according as Ω is positive or negative.

Combining equations (5) and (7), and remembering that

$$\operatorname{sgn} \Phi(x_2) = \operatorname{sgn} \Phi(x_1 + \epsilon),$$

we get:

$$\operatorname{sgn} \Psi(x_2)$$
$$= i \cdot \operatorname{sgn} (\phi_1\psi_2 - \phi_2\psi_1) \cdot \operatorname{sgn} (\psi_1\chi_2 - \psi_2\chi_1) \cdot \operatorname{sgn} (\chi_1\phi_2 - \chi_2\phi_1) \cdot \operatorname{sgn} \Psi(x_1).$$

This equation shows that $\Psi(x_2)$ and $\Psi(x_1)$ have opposite signs. Therefore Ψ vanishes between x_1 and x_2—as was to be proved.

We have thus proved the first statement of our theorem and the proof of the other two statements follows at once from considerations of symmetry.

The case in which the product of the six functions considered in the last theorem is positive can be reduced to the case just considered by interchanging two of the three letters ϕ, ψ, χ. If, further, we remember that under the conditions of the last theorem the roots of any two of the functions Φ, Ψ, X separate each other (cf. theorem III) we obtain the following theorem:

VII. *If none of the six functions*

$$\phi_1\psi_2 - \phi_2\psi_1, \quad \psi_1\chi_2 - \psi_2\chi_1, \quad \chi_1\phi_2 - \chi_2\phi_1, \quad \{\phi_1, \phi_2\}, \quad \{\psi_1, \psi_2\}, \quad \{\chi_1, \chi_2\}$$

vanish in (I), *and if the last three of them have the same sign, then the roots of the functions* Φ, Ψ, X *follow each other cyclically in the order named or in the reverse order* X, Ψ, Φ *according as the product of the six functions above written is negative or positive.*

As an application of this theorem let us consider the three functions:

$$\Phi = y, \quad \Psi = y', \quad \mathrm{X} = py' + qy = -y''.$$

We must here assume that the coefficients p, q of the differential equation have finite first derivatives at every point of (I). We have in this case:

$$\phi_1\psi_2 - \phi_2\psi_1 = -1, \quad \psi_1\chi_2 - \psi_2\chi_1 = q, \quad \chi_1\phi_2 - \chi_2\phi_1 = p, \quad \{\phi_1, \phi_2\} = 1,$$

$$\{\psi_1, \psi_2\} = q, \quad \{\chi_1, \chi_2\} = qp' - q'p + q^2 = q^2\left[\frac{d}{dx}\left(\frac{p}{q}\right) + 1\right].$$

VIII. *If throughout the interval* (I):

$$q > 0, \quad \frac{d}{dx}\left(\frac{p}{q}\right) > -1,$$

and p *does not vanish in this interval, then the roots of* y, y', y'' *follow each other in the cyclic order named if* $p > 0$, *in the reverse order if* $p < 0$.

As an application of this theorem let us consider the positive roots of the three Bessel's functions $J_n(x), J_{n+1}(x), J_{n+2}(x)$ where n has any real value. Let us introduce the new independent variable $\xi = x^2/4$ and let

$$R_m(\xi) = \left(\frac{2}{x}\right)^m J_m(x).$$

Then we have:*

$$R'_m(\xi) = -R_{m+1}(\xi).$$

The relative position of the positive roots of J_n, J_{n+1}, J_{n+2} is then evidently the same as that of the positive roots of R_n, R'_n, R''_n. Now R_n satisfies the differential equation:

$$y'' + \frac{n+1}{\xi} y' + \frac{1}{\xi} y = 0.$$

By applying theorem VIII to this equation we get at once the theorem:

IX. *The positive roots of J_n, J_{n+1}, J_{n+2} follow each other cyclically in the order named if $n > -1$, in the reverse order if $n < -1$.*†

§ 3. *An extension of the previous results.*

We shall begin by considering the non-homogeneous linear equation of the first order.

$$y' + py = r, \tag{8}$$

where p and r are throughout the interval (I) real and continuous functions of x.

It is well known that there exists one and only one solution of this equation which vanishes at an arbitrary point c of (I) and that this solution is given by the formula:

$$y = e^{-\int_c^x p dx} \int_c^x r e^{\int_c^x p dx} dx.$$

This formula shows us that if throughout (I) $r \geqq 0$ the following relations are satisfied:

$$y \geqq 0 \qquad (c < x \leqq b),$$

$$y \leqq 0 \qquad (c > x \geqq a).$$

* Cf. GRAY and MATHEWS: *Treatise on Bessel Functions*, p. 46.

† This theorem was established by the writer in the Bulletin of the American Mathematical Society, March, 1897, p. 207, although owing to an oversight the reversal of the order of the roots when $n < -1$ is not mentioned there. The part of this theorem which states that the roots of J_n and J_{n+1} separate each other is of earlier date, it having been stated without proof by GRAY and MATHEWS (l. c.) as being probably correct, and actually proved by VAN VLECK (American Journal of Mathematics, vol. 19 (1897), p. 75). Among a considerable number of other proofs which have since been given of this part of the theorem I should like to call special attention to an extremely brief and simple one by PORTER (American Journal of Mathematics, vol. 20 (1898), p. 199, and Bulletin of the American Mathematical Society, March, 1898, p. 274) which has recently been reproduced by H. WEBER in his edition of RIEMANN'S *Differentialgleichungen der mathematischen Physik*, vol. 1 (1900), p. 166.

If on the other hand $r \leqq 0$ these last two inequalities must be reversed. Moreover it is clear that in each of these cases y will vanish at a point x when and only when r vanishes at all points between c and x.

X. *If r does not change sign in (I) a solution of (8) vanishes either at no point of (I), at one and only one point, or at all the points of a connected portion of (I) throughout which $r = 0$, but at no other point of (I). In the last two cases:*

if $r \geqq 0$ $\qquad y(a) \leqq 0, \quad y(b) \geqq 0,$

if $r \leqq 0$ $\qquad y(a) \geqq 0, \quad y(b) \leqq 0.$*

Let us now consider the very much more general differential equation:

$$y' + p(x, y) \cdot y = r(x, y), \tag{9}$$

where p and r are continuous real functions of the two independent variables (x, y) so long as x remains in (I), y having any value.† Let us suppose that y_1 is a solution of (9) which is continuous throughout (I). Then y_1 is also a solution of the linear equation:

$$y' + p(x, y_1) \cdot y = r(x, y_1),$$

where p and r are now continuous functions of x only. Moreover, if $r(x, y)$ does not change sign the same will be true *a fortiori* of $r(x, y_1)$. Accordingly in this case *all that is said in theorem* X *about y is true of every solution of* (9) *which is continuous throughout (I).*

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* I should like to indicate in passing how this theorem can be used to prove the theorems contained on pages 415, 416 in my paper in vol. 1 of these Transactions.

We had there two functions ω_1 and ω_2 continuous throughout (I) and satisfying respectively the RICCATI'S equations:

$$\omega' = A_1 + C_1\omega^2,$$
$$\omega' = A_2 + C_2\omega^2,$$

where the functions A_1, A_2, C_1, C_2 are continuous and satisfy throughout (I) the inequalities:

$$A_2 \geqq A_1, \quad C_2 \geqq C_1.$$

It is clear that the difference $\omega_2 - \omega_1$ satisfies the linear equation:

$$y' - C_2(\omega_2 + \omega_1)y = (A_2 - A_1) + (C_2 - C_1)\omega_1^2.$$

Since $A_2 - A_1 + (C_2 - C_1)\omega_1^2 \geqq 0$, theorem X shows that if $\omega_2(a) > \omega_1(a)$ then $\omega_2(x) > \omega_1(x)$, while if $\omega_2(a) = \omega_1(a)$ then $\omega_2(x) \geqq \omega_1(x)$, and in fact $\omega_2(x) > \omega_1(x)$ provided we exclude the special cases for which throughout the neighborhood of a, $A_2 = A_1$ and $(C_2 - C_1)\omega_1^2 = 0$.

While the proof here given is perhaps a little more artificial than the one I used before, it has the advantage, besides being slightly shorter, of dispensing with the use of theorems of a not altogether elementary character about differential equations involving parameters.

† The more general case in which p and r are continuous only when y is restricted to a finite interval might easily be considered.

In particular the RICCATI's equation

$$\omega' = A + B\omega + C\omega^2, \tag{10}$$

where A, B, C are continuous real functions of x throughout (I), is of the form (9) if we let $r = A$.

XI. *If the function A does not change sign in (I) and does not vanish at all points of a continuous portion of (I), than a real solution ω of* (10) *which is continuous throughout (I) cannot vanish at more than one point of (I), and if it does vanish it passes with increasing x from negative to positive if $A \geqq 0$, from positive to negative if $A \leqq 0$.*

By applying this theorem to equation (2) we get the following generalization of theorem II:

XII. *If $\phi_1\psi_2 - \phi_2\psi_1$ does not vanish in (I), and $\{\phi_1, \phi_2\}$ does not change sign and does not vanish at all points of a continuous portion of (I), and if $\phi_1', \phi_2', \psi_1', \psi_2'$, are continuous throughout (I), then in any portion of (I) in which Ψ does not vanish Φ cannot vanish more than once.*

From this we deduce a generalization of theorem I:

XIII. *If ϕ_1 and ϕ_2 do not both vanish at any point of (I) and if $\{\phi_1, \phi_2\}$ does not change sign in (I) and does not vanish at all points of a continuous portion of (I), and if ϕ_1', ϕ_2' are continuous throughout (I), then Φ cannot vanish more than a finite number of times in (I), and when it vanishes it changes sign.*

It should be noticed that it is no longer true that Φ and Φ' cannot vanish together. In a similar manner the other theorems of §§ 1, 2 can be generalized by replacing the requirement that the functions $\{\phi_1, \phi_2\}$, $\{\psi_1, \psi_2\}$, $\{\chi_1, \chi_2\}$ do not vanish, by the requirement that they do not change sign and do not vanish at all points of a continuous portion of (I), and adding the requirement that ϕ_1', ϕ_2', ψ_1', ψ_2', χ_1', χ_2' are continuous. The proofs of some of the theorems must, however, be slightly modified. In particular, formulæ (3) and (4) must be replaced by the following:

At a point x_1 where $\Phi = 0$:

$$\operatorname{sgn}\frac{\Delta\Phi}{\Delta x} = \operatorname{sgn}\Phi(x_1+\epsilon) = -\operatorname{sgn}\Phi(x_1-\epsilon)$$
$$= -\operatorname{sgn}\{\phi_1, \phi_2\}\cdot\operatorname{sgn}(\phi_1\psi_2 - \phi_2\psi_1)\cdot\operatorname{sgn}\Psi(x_1).$$

At a point x_2 where $\Psi = 0$:

$$\operatorname{sgn}\frac{\Delta\Psi}{\Delta x} = \operatorname{sgn}\Psi(x_2+\epsilon) = -\operatorname{sgn}\Psi(x_2-\epsilon)$$
$$= \operatorname{sgn}\{\psi_1, \psi_2\}\cdot\operatorname{sgn}(\phi_1\psi_2 - \phi_2\psi_1)\cdot\operatorname{sgn}\Phi(x_2).$$

GRUND IM HARZ,
July 20, 1901.

ON THE SYSTEM OF A BINARY CUBIC AND QUADRATIC AND THE REDUCTION OF HYPERELLIPTIC INTEGRALS OF GENUS TWO TO ELLIPTIC INTEGRALS BY A TRANSFORMATION OF THE FOURTH ORDER*

BY

JOHN HECTOR McDONALD

INTRODUCTION.

1. If a hyperelliptic integral of genus 2 is reducible to an elliptic integral by a transformation of order k then there always exists a second integral † with the same irrationality which is reducible by a transformation of the same order. In the algebraic treatment of the problem the difficulty lies in the determination of the second integral after the first has been constructed.

For $k = 4$ the problem has been treated by Professor BOLZA ‡ and in the sixth section of his dissertation the formulæ for the two reducible integrals are established algebraically. Professor BOLZA used a particular system of variables x_1, x_2. These are chosen so that if

$$y_1 = a_0x_1^4 + 4a_1x_1^3x_2 + 6a_2x_1^2x_2^2 + 4a_3x_1x_2^3 + a_4x_2^4$$

and

$$y_2 = b_0x_1^4 + 4b_1x_1^3x_2 + 6b_2x_1^2x_2^2 + 4b_3x_1x_2^3 + b_4x_2^4$$

are the two forms of order 4 which are equal to the homogeneous variables in the elliptic integral, then $a_1 = 0$ and $b_1 = 0$, and the numerators of the two reducible integrals turn out to be x_1 and x_2. The derivation of the second integral depends on the use of this particular system of variables and the connection of these variables with the irrationality and with the reducing substitutions is unexplained.

* Presented to the Society (Chicago) April 6, 1901, under a slightly different title. Received for publication May 17, 1901.

† This is a consequence of the WEIERSTRASS-PICARD Theorem: Acta Mathematica, vol. 4 (1884), p. 400; Bulletin de la Société Mathématique de France, vol. 11 (1882–1883), p. 25.

‡ *Inaugural Dissertation*, Göttingen, 1886; Mathematische Annalen, vol. 28 (1887), p. 447.

The object of this paper is to supply the following desiderata: (*a*) a solution of the problem independently of a special system of variables, by using theorems on biquadratic involutions, (*b*) a methodical deduction of the second integral from the first, (*c*) the introduction of the system of variables and the normal form which are appropriate to the problem.

The integral

$$\int \frac{(ax+b)\,dx}{\sqrt{R_6(x)}}$$

is a hyperelliptic integral of the first kind of genus 2 if $R_6(x)$ is a polynomial of the 6th degree, and it is reducible to an elliptic integral by a transformation of order 4 if

$$\int \frac{(ax+b)\,dx}{\sqrt{R_6(x)}} = M\int \frac{dy}{\sqrt{R_4(y)}},$$

where $R_4(y)$ is a polynomial of order 3 or 4 and $y = U(x)/V(x)$, U and V being polynomials, the higher of whose orders is 4 and M a multiplier not depending on x or y.

In homogeneous variables $x = x_1/x_2$, $y = y_1/y_2$ and, with the usual notations, the above relation becomes

$$\int \frac{(xd)(x\,dx)}{\sqrt{R_6(x_1x_2)}} = M\int \frac{(y\,dy)}{\sqrt{R_4(y_1y_2)}},$$

where

$$y_1 = \rho\, U(x_1x_2),$$
$$y_2 = \rho\, V(x_1x_2),$$
$$(xd) = x_1d_2 - x_2d_1.$$

Denoting more explicitly $R_6(x_1x_2)$ by $(x\alpha)(x\beta)(x\gamma)(x\delta)(x\kappa)(x\lambda)$ and $R_4(y_1y_2)$ by $(y\mu)(y\nu)(y\pi)(y\omega)$ we see by reasoning similar to that used by JACOBI* for the transformation of elliptic integrals that it is possible to break up R_6 into quadratic factors $(x\alpha)(x\beta)$, $(x\gamma)(x\delta)$, $(x\kappa)(x\lambda)$ and to determine linear forms (xd'), (xd''), (xd''') and a quadratic form $(x\xi)(x\eta)$ so that

$$\mu_2U - \mu_1V = (x\alpha)(x\beta)(xd')^2,$$
$$\nu_2U - \nu_1V = (x\gamma)(x\delta)(xd'')^2,$$
$$\pi_2U - \pi_1V = (x\kappa)(x\lambda)(xd''')^2,$$
$$\omega_2U - \omega_1V = (x\xi)^2(x\eta)^2;$$

and if we denote by ϑ the Jacobian of U and V then

$$\vartheta = (xd)(xd')(xd'')(xd''')(x\xi)(x\eta).$$

* JACOBI, *Fundamenta Nova*, *Werke*, vol. 1, p. 57.

Conversely, if these relations are satisfied, the integral

$$\int \frac{(xd)(x\,dx)}{\sqrt{R_6(x_1x_2)}}$$

is reducible by the substitution $y_1 : y_2 = U : V$; that is, these conditions are sufficient as well as necessary.

Accordingly with every reducible hyperelliptic integral is associated a biquadratic involution $I_4 = \lambda U + \nu V$ which is special since it contains a complete square, viz., $(x\xi)^2(x\eta)^2$. Every such involution contains, aside from the square $(x\xi)^2(x\eta)^2$, four forms having double roots. The quadratic factor which goes along with the double factor to make up any one of these four forms is called a branch quadratic and the forms themselves branch forms. It follows from the above equations that the product of three branch quadratics of an involution containing a square gives a sextic belonging to a reducible integral and that the numerator is furnished by the double element in the fourth branch form. To every reducible integral belongs an involution I_4 in which one of the four branch forms is distinguished from the other three, each of which plays the same rôle.

In section I it is shown that there exists one and only one I_4 which contains the square of an arbitrary quadratic f, and has the factors of an arbitrary cubic ϕ for three of its double elements. An I_4 determined in such a way has evidently one branch form distinguished from the rest and the three which contain the factors of ϕ are not distinguished from each other.

In sections II, III it is shown that if ϕ, f be a system (Σ) of a cubic and a quadratic there exists a covariant system (Σ) of a cubic and a quadratic ϕ, f, such that ϕ, f are the same covariants of $\bar{\phi}, \bar{f}$ as ϕ, f are of ϕ, f, or in other words, the relation between (Σ) and (Σ) is mutual. The forms ϕ and f are obtained as follows. Using transvectant notation put $\phi = l_x m_x n_x$, the product of linear factors, and let p and q be two linear forms and also let

$$D_{pq}(\phi) = 4^3(p^3q,\ l)_1(p^3q,\ m)_1(p^3q,\ n)_1 p^{-6}.$$

Then $D_{pq}(\phi)$ is an integral covariant of ϕ, p, q. If we let p and q be the two linear covariants of ϕ and f so denoted in the notation of CLEBSCH (*Binäre Formen**) $D_{pq}(\phi)$ will be a covariant of ϕ and f. Also let $Q(\phi)$ be the cubic covariant of ϕ, $D_{pq}\big(Q(\phi)\big) = \phi' = c_x^3$ and $f'' = d_x^2 =$ the Jacobian of pq and $(cq)c_x^2$. Further, let

$$\sigma = \frac{(cd)^2 c_x}{q}, \quad \rho = \frac{(cd)^2(cd')\,d'_x}{p};$$

σ and ρ are invariants of ϕ, f. In these notations the following theorem holds.

* If $\phi = a_x^3$, $f = b_x^2$, $p = (ab)^2 a_x$, $q = (ab)^2(ab')b'_x$.

THEOREM: *The forms*

$$\frac{\rho}{\sigma^2}\phi' = \bar{\phi}, \quad \frac{\sigma}{\rho}f' = \bar{f}$$

are covariants of ϕ, f *and if* $\bar{\phi}, \bar{f}$ *are the same covariants of* $\phi, \bar{f}$ *as* ϕ, f *were of* ϕ, f *then*

$$\bar{\bar{\phi}} = \phi, \quad \bar{\bar{f}} = f.$$

Also, if the covariants $\bar{p}, \bar{q}$ *are formed from* $\bar{\phi}$ *and* $\bar{f}$*;*

$$\bar{p} = q, \quad \bar{q} = p.$$

It is on the existence of these mutual relations between the systems (Σ) and $(\bar{\Sigma})$ that the second reducible integral depends.

In section IV the involutions I_4 and $\bar{I}_4$ belonging to (Σ) and $(\bar{\Sigma})$ are determined and it is shown that the Jacobian of $I_4 = \phi f q$ and of $\bar{I}_4 = \bar{\phi}\bar{f}\bar{q} = \bar{\phi}\bar{f}p$; moreover the three branch quadratics belonging to the three branch forms whose double elements are the factors of ϕ are identical with those in $\bar{I}_4$ whose double elements are the factors of $\bar{\phi}$, except as to sign. If these branch quadratics are denoted by R, S, T then

$$\int \frac{q(x\,dx)}{\surd RST}, \quad \int \frac{p(x\,dx)}{\surd RST}$$

are both reducible integrals with the same irrationality, the first by I_4, the second by $\bar{I}_4$. The statement of this theorem constitutes section V.

In section VI, the variables $x_1' = q$, $x_2' = p$ are introduced as leading to an appropriate canonical form, which is identical with that used by Professor BOLZA in his solution of the problem.

In section VII a number of miscellaneous results are collected relating to biquadratic involutions containing a complete square and their geometrical representations on a rational quartic and a twisted cubic.

I.

THEOREMS ON THE BIQUADRATIC INVOLUTION HAVING A COMPLETE SQUARE.

2. *A biquadratic involution* I_4 *containing a square is determined without ambiguity by the quadratic form whose square it contains and by any three of its other double elements, or in other words, by a cubic and a quadratic. These forms are independent and their introduction singles out one of the branch forms of the involution from the others.*

Before proving these statements another result must be proved.

LEMMA.—*Let* U *and* f^2 *be two biquadratic forms. Let the Jacobian of* U

and f^2, which must contain f, be $f\theta$. The form θ is apolar to the forms of the involution $\lambda U + \mu f^2$.*

Proof.—Let U and f be taken in a normal form found by introducing as variables the linear factors of f. Then

$$f = x_1 x_2$$

and

$$U = a_0 x_1^4 + 4a_1 x_1^3 x_2 + 6a_2 x_1^2 x_2^2 + 4a_3 x_1 x_2^3 + a_4 x_2^4.$$

The Jacobian of f^2, U is found on calculation to be:

$$\tfrac{1}{2} x_1 x_2 (a_0 x_1^4 + 2a_1 x_1^3 x_2 - 2a_3 x_1 x_2^3 - a_4 x_2^4),$$

and so

$$\theta = \tfrac{1}{2} (a_0 x_1^4 + 2a_1 x_1^3 x_2 - 2a_3 x_1 x_2^3 - a_4 x_2^4).$$

Then on calculation one finds the desired relations

$$(u, \theta)_4 = 0, \qquad (f^2, \theta)_4 = 0.$$

3. We now prove the following theorem:

THEOREM.—*Let ϕ denote any cubic and f any quadratic not apolar to ϕ. There exists one and but one I_4 containing the form f^2 and having three of its double elements given by $\phi = 0$.*

Proof.—We can always determine a linear form $(x\delta)$ so that $\phi(x\delta)$ is apolar to f^2, and in but one way. For $(\phi(x\delta) f^2)_1 \not\equiv 0$, since f^2 is not apolar to ϕ;† hence $(\phi(x\delta) f^2)_4 = 0$ is an equation of the first degree to determine δ_1/δ_2.

Let $f = x_1 x_2$, the factors of f being introduced as variables; then we may take

$$\phi(x\delta) = c_0 x_1^4 + 4c_1 x_1^3 x_2 + 6c_2 x_1^2 x_2^2 + 4c_3 x_1 x_2^3 + c_4 x_2^4;$$

but $c_2 = 0$, since $\phi(x\delta)$ is apolar to $f^2 = x_1^2 x_2^2$, hence

$$\phi(x\delta) = c_0 x_1^4 + 4c_1 x_1^3 x_2 + 4c_3 x_1 x_2^3 + c_4 x_2^4,$$
$$f^2 = x_1^2 x_2^2.$$

First. It appears at once that there is one I_4 satisfying the required conditions, viz.:

$$I_4 = \lambda x_1^2 x_2^2 + \mu (c_0 x_1^4 + 8c_1 x_1^3 x_2 - 8c_3 x_1 x_2^3 - c_4 x_2^4).$$

For calculating the form θ we find it is equal to

$$\tfrac{1}{2} (c_0 x_1^4 + 4c_1 x_1^3 x_2 + 4c_3 x_1 x_2^3 + c_4 x_2^4),$$

hence I_4 contains f^2 and ϕ gives three of its double elements.

---

* Two forms $f = a_x^n$, $\phi = b_x^n$ are apolar if

$$(f, \phi)_n = (ab)^n = \sum_{i=0 \cdots n} a_i b_{n-i} (-1)^i {}_n C_i = 0.$$

† If ϕ and f are apolar, that is, if $(\phi f)_2 \equiv 0$, then $\phi(x\delta)$ is apolar to f^2 whatever $(x\delta)$ may be.

Second. There can be only one such I_4. For suppose another to exist:

$$I_4' = \lambda x_1^2 x_2^2 + \mu(c_0' x_1^4 + 8c_1' x_1^3 x_2 - 8c_3' x_1 x_2^3 - c_4' x_2^4),$$

we should have

$$x_1 x_2 \theta = \sigma x_1 x_2 \theta',$$

since the Jacobians of I_4 and I_4' must both be $\phi(x\delta)f$.

Hence

$$c_0' x_1^4 + 4c_1' x_1^3 x_2 + 4c_3' x_1 x_2^3 + c_4' x_2^4 = \sigma(c_0 x_1^4 + 4c_1 x_1^3 x_2 + 4c_3 x_1 x_2^3 + c_4 x_2^4),$$

or

$$c_0' = \sigma c_0, \quad c_1' = \sigma c_1, \quad c_3' = \sigma c_3, \quad c_4' = \sigma c_4;$$

or

$$c_0' x_1^4 + 8c_1' x_1^3 x_2 - 8c_3' x_1 x_2^3 - c_4' x_2^4 = \sigma(c_0 x_1^4 + 8c_1 x_1^3 x_2 - 8c_3 x_1 x_2^3 - c_4 x_2^4);$$

that is

$$c_0' x_1^4 + 8c_1' x_1^3 x_2 - 8c_3' x_1 x_2^3 - c_4' x_2^4$$

belongs to I_4; hence I_4' is identical with I_4, which is contrary to hypothesis.

In virtue of this theorem we may consider a special I_4 containing a square as determined uniquely by a cubic and quadratic ϕ and f.

4. The following theorem determines $(x\delta)$ more exactly. In symbolic notation let $\phi = a_x^3, f = b_x^2$.

THEOREM: *$(x\delta)$ is proportional and may be taken equal to $(ab)^2(ab')b_x'$.*

To prove this we must show that

$$\left(a_x^3 \cdot (a'b)^2(a'b')b_x', \quad b_x''^2 b_x'''^2\right)_4 = 0.$$

Proof.—We have

$$(a_x^3, b_x''^2 b_x'''^2)_3 = \tfrac{4}{3}(ab'')^2(ab''')b_x''',$$

and therefore

$$\left((ab'')^2(ab''')b_x''', \quad (a'b)(a'b')b_x'\right)_1$$

must be zero, or

$$(ab'')^2(ab''')(a'b)(a'b')(b'b''') = 0.$$

If we interchange b', b'''; a, a'; b, b'', which has no effect on the value of the expression, we find

$$(ab'')^2(ab''')(a'b)(a'b')(b'b''') = (ab'')^2(ab''')(a'b)(a'b')(b'''b')$$
$$= -(ab'')^2(ab''')(a'b)(a'b')(b'b'''),$$

or

$$(ab'')^2(ab''')(a'b)(a'b')(b'b''') = 0.$$

II.

The System of a Cubic and two Linear Forms and their Conjugate System.

5. We consider next a system of a cubic form ϕ resolved into its linear factors, $\phi = l_x m_x n_x$, and two linear forms p_x and q_x (denoting them for brevity by l, m, n, p, q), returning later to the system of a cubic and quadratic. The discussion will be much facilitated by three theorems.

Let

$$D_{pq}(\phi) = 4^3 (p^3q, l)_1 (p^3q, m)_1 (p^3q, n)_1 p^{-6}.$$

Here $D_{pq}(\phi)$ is an integral covariant of ϕ, p, q since $(p^3q, l)_1$ is divisible by p^2.

THEOREM I: *If*

$$D_{pq}(\phi) = 4^3 (p^3q, l)_1 (p^3q, m)_1 (p^3q, n)_1 p^{-6},$$

where l, m, n are the linear factors of ϕ, then

$$D_{qp}\big(D_{pq}(\phi)\big) = 27\,(pq)^6\phi.$$

Proof.—We have

$$4\,(p^3q, l)_1 = p^2\big(p(ql) + 3q(pl)\big),$$

and find that

$$D_{pq}(\phi) = \prod_{l,\,m,\,n}\big(p(ql) + 3q(pl)\big) = l'_x m'_x n'_x = l'm'n', \quad 4\,(q^3p, l')_1 = 3\,(pq)^2 q^2 l$$

by a short computation, making use of the identity

$$l_x(pq) + p_x(ql) + q_x(lp) = 0.$$

Hence

$$D_{qp}(l'm'n') = D_{qp}\big(D_{pq}(\phi)\big) = 27\,(pq)^6\phi.$$

THEOREM II: *Let $Q(\phi)$ denote the cubic covariant of ϕ; then*

$$Q\big(D_{pq}(\phi)\big) = -\,27\,(pq)^6 D_{pq}\big(Q(\phi)\big).$$

Let

$$L = (ml)n_x + (nl)m_x = 2(l, nm)_1,$$

$$M = (nm)l_x + (lm)n_x = 2(m, ln)_1,$$

$$N = (ln)m_x + (mn)l_x = 2(n, lm)_1,$$

so that Ll is harmonic to mn. Then $LMN = \sigma Q$, since the cubic covariant of a cubic may be found by taking the conjugate of each factor with respect to the other two and forming the product. The factor σ is a numerical constant since LMN and Q are both of the 3d degree in the coefficients of ϕ; this appears at once from the expressions of L, M, N in l, m, n.

We have

$$D_{pq}\big(Q(\phi)\big) = D_{pq}(\sigma LMN) = \sigma \prod_{L,M,N} \big(p(qL) + 3q(pL)\big)$$

$$= \sigma \prod_{l,m,n} p\{(ml)(qn) + (nl)(qm)\} + 3q\{(ml)(pn) + (nl)(pm)\},$$

and

$$2(l', m'n') = \big(p(qm) + 3q(pm)\big)\big((pl')(qn) + 3(pn)(ql')\big)$$

$$+ \big(p(qn) + 3q(pn)\big)\big((pl')(qm) + 3(pm)(ql')\big);$$

but

$$(pl') = 3(pl)(pq), \quad (ql') = (qp)(qm).$$

Substituting in $(l', m'n')_1$, and making use of the relation

$$(pl)(qn) + (pq)(nl) + (pn)(lq) = 0,$$

derivable from $l_x(qn) + q_x(nl) + n_x(lq) = 0$ by putting $x_1 : x_2 = -p_2 : p_1$, we find

$$2(l', m'n')_1 = -3(pq)^2\big\{p\big((ml)(qn) + (nl)(qm)\big) + 3q\big((ml)(pn) + (nl)(pm)\big)\big\}.$$

Put

$$\phi' = l'm'n';$$

then

$$Q(\phi') = \sigma 2^3 (l', m'n')_1 (m', l'n')_1 (n', m'l')_1$$

$$= \sigma \prod_{l,m,n} -3(pq)^2\big\{p\big((ml)(qn) + (nl)(qm)\big)$$

$$+ 3q\big((ml)(pn) + (nl)(pm)\big)\big\}$$

$$= -27(pq)^6 D_{pq}\big(Q(\phi)\big),$$

or

$$Q\big(D_{pq}(\phi)\big) = -27(pq)^6 D_{pq}\big(Q(\phi)\big).$$

6. THEOREM III: *Let R be the discriminant of ϕ; then*

$$D_{qp}\big(Q(\phi')\big) = -\frac{R^2}{4} 27^2 (pq)^{12}\phi.$$

Proof.—By definition

$$D_{qp}\big(Q(\phi')\big) = D_{qp}\big(Q\{D_{pq}[Q(\phi)]\}\big);$$

but we may interchange the two inside operators by theorem II; therefore

$$D_{qp}\big(Q(\phi')\big) = -27(pq)^6 D_{qp}\big(D_{pq}\{Q[Q(\phi)]\}\big);$$

the factor $-27(pq)^6$ may be taken out because $D_{qp}(\phi)$ is linear in the coefficients of ϕ, viz., $D_{qp}(a\phi) = aD_{qp}(\phi)$. By theorem I,

$$D_{qp}\big(D_{pq}\{Q[Q(\phi)]\}\big) = 27(pq)^6 Q\big(Q(\phi)\big);$$

hence

$$D_{qp}\big(Q(\phi')\big) = -27^2(pq)^{12}Q\big(Q(\phi)\big),$$

and

$$Q\big(Q(\phi)\big) = -4^{-1}R^2\phi$$

(CLEBSCH, *Binäre Formen*); therefore

$$D_{qp}\big(Q(\phi')\big) = -4^{-1}R^2 27^2(pq)^{12}\phi.$$

III.

THE SYSTEM OF A CUBIC AND QUADRATIC AND THEIR CONJUGATE SYSTEM.

7. It is necessary now to recall two properties of the form system of a cubic and quadratic c_x^3 and d_x^2: 1°. The first polar of c_x^3 with respect to the linear covariant $(cd)^2c_x$ of c_x^3 and d_x^2 is a quadratic harmonic to d_x^2; 2°. The linear covariant $(cd)^2(cd')d_x'$ is the polar of $(cd)^2c_x$ with respect to d_x^2, or the product of these two linear covariants is harmonic to d_x^2.

Proof.—1°. Let (xy) be any linear form; then $c_x^2c_y$ is harmonic to d_x^2 if $(c_x^2c_y, d_x^2)_1 = 0$ or $(cd)^2c_y = 0$; hence $(xy) = (cd)^2c_x$.

2°. If we form $\big(d_x^2, (cd)^2c_x\big)_1$ we find $(cd)^2(cd')d_x'$. Hence the truth of the above statements.

8. Consider the form $\phi' = D_{pq}\big(Q(\phi)\big) = c_x^3$ and also $(c_x^3, q)_1 = (cq)c_x^2$. Then there is a quadratic form $f' = d_x^2$ which is the Jacobian of $(cq)c_x^2$ and pq. Then from the first property of the previous paragraph $(cd)^2c_x = \sigma q$ and $(cd)^2(cd')d_x' = \rho p$, where σ and ϕ are proportionality factors,* for q and p have just the properties of the two linear covariants; which properties define them except as to multipliers.

Further, it follows that ρ and σ are invariants of the system ϕ, p, q. For ϕ' is a covariant of this system and so consequently is also $f' = d_x^2$, and therefore $(cd)^2c_x$, $(cd)^2(cd')d_x'$, and finally, $(cd)^2c_xq^{-1} = \sigma$, $(cd)^2(cd')d_xp^{-1} = \rho$.

Take then the covariants $\rho\sigma^{-2}\phi' = C_x^3 = \bar{\phi}$, $\sigma\rho^{-1}f' = D_x^2 = f$ of ϕ, p, q; then $(CD)^2C_x = \rho\sigma^{-2}\cdot\sigma\rho^{-2}(cd)^2c_x = q$ and

$$(CD)^2(CD')D_x' = \rho\sigma^{-2}\cdot\sigma^2\rho^2\cdot(cd)(cd')d_x' = p.$$

Hence q and p are these two linear covariants of $\bar{\phi}, f$.

From the reasoning just used it is clear that if ϕ' is any cubic and q, p any two linear forms there is but one solution to the problem to find a quadratic f and a multiplier k so that the two linear covariants defined above of $k\phi$ and f should be q and p.

* We determine σ and ρ in § 11.

For let $\bar{\phi}$ and $\bar{f}$ be one solution, $\bar{\phi} = C_x^3, \bar{f} = D_x^2$; and let $u\bar{\phi} = \Gamma_x^3$, $v\bar{f} = \Delta_x^2$ be any solution. Then

$$(\Gamma\Delta)^2\Gamma_x = (CD)^2 C_x, \quad (\Gamma\Delta)^2(\Gamma\Delta')\Delta'_x = (CD)^2(CD')D'_x,$$

or

$$uv(CD)^2 C_x = (CD)^2 C_x, \quad uv^2(CD)^2(CD')D'_x = (CD)^2(CD')D'_x;$$

hence $uv = 1$, $uv^2 = 1$, the only solution of which is $u = 1$, $v = 1$, which was to be proved.

This remark will be useful in the sequel.

9. Suppose given a system (Σ) of a cubic ϕ and quadratic f, in symbolic notation a_x^3, b_x^2; let $(ab)^2 a_x = p$ and $(ab)^2(ab')b'_x = q$ where ϕ, p, q are to be used as in the preceding paragraphs. Let $\bar{\phi}, \bar{f}$ be formed according to the prescription given there, calling the system of ϕ, f, (Σ). Then $\bar{\phi}, \bar{f}$ are two covariants of (Σ).

THEOREM.—*If we form the same two covariants* $\bar{\bar{\phi}}, \bar{\bar{f}}$, *of* (Σ) *as* $\bar{\phi}, \bar{f}$ *are of* (Σ) *then*

$$\bar{\bar{\phi}} = \phi, \quad \bar{\bar{f}} = f;$$

that is (Σ) *may be derived from* (Σ) *as* $(\bar{\Sigma})$ *was from* (Σ).

Proof.—Remembering that the covariants of (Σ) which we call p, q are for $(\bar{\Sigma})$ equal to q, p we operate with ϕ, q, p to produce $\bar{\bar{\phi}}$ as we operated with ϕ, p, q to produce $\bar{\phi}$.

Since $Q(a\phi) = a^3 Q(\phi)$ and $D_{pq}\big(a^3 Q(\phi)\big) = a^3 D_{pq}\big(Q(\phi)\big)$ and $\bar{\phi} = \rho\sigma^{-2}\phi'$ it follows that

$$D_{qp}\big(Q(\phi)\big) = \frac{\rho^3}{\sigma^6} D_{qp}\big(Q(\phi')\big) = -\frac{R^2}{4} 27^2 (pq)^2 \frac{\rho^3}{\sigma^6}\phi = \phi''.$$

From ϕ'', p, q deduce $\bar{\bar{\phi}}, \bar{\bar{f}}$ as ϕ, f were deduced from ϕ', q, p. Then $\bar{\bar{\phi}}$ is proportional to ϕ'' and if $\bar{\bar{\phi}} = A_x^3$, $\bar{\bar{f}} = B_x^2$, we have

$$(AB)^2 A_x = p, \quad (AB)^2(AB')B'_x = q.$$

But ϕ is proportional to ϕ'' and $\phi = a_x^3$, $f = b_x^2$ have the property $(ab)^2 a_x = p$, $(ab)^2(ab')b'_x = q$; therefore, by the remark at the end of § 8,

$$\bar{\bar{\phi}} = \phi, \bar{\bar{f}} = f.$$

We have then *complete reciprocity* between two systems of a cubic and quadratic (Σ) and $(\bar{\Sigma})$. If we denote forms derived analogously from the two systems by a stroke we have the scheme:

(Σ)	$(\bar{\Sigma})$
ϕ, f	$\bar{\phi}, \bar{f}$
p, q	$\bar{p}, \bar{q}$

and $p = q$, $\bar{q} = p$.

This duality has some resemblance to the relations existing between a binary cubic and its cubic covariant, the pencil of a binary biquadratic and its hessian and the hessians of the pencil,* and the pencil of a ternary cubic and its hessian and the hessians of the pencil.†

10. Let $\psi, \phi_1, \cdots \phi_n$ denote any covariants of (Σ) and $\psi, \phi_1, \cdots \phi_n$ the corresponding covariants of (Σ) and suppose we have a relation

$$\chi\left((\psi\bar{\psi})_\lambda, \phi_1 \cdots \phi_n\right) = 0;$$

then because of the duality of (Σ), (Σ) we have also

$$\chi\left(\pm(\psi\bar{\psi})_\lambda, \bar{\phi}_1 \cdots \bar{\phi}_n\right) = 0,$$

since

$$(\bar{\psi}\psi)_\lambda = \pm(\psi\bar{\psi})_\lambda,$$

according as λ is even or odd.

By means of such a relation we are able to show the duality which exists between the involutions I_4, $\bar{I}_4$ determined by (Σ), $(\bar{\Sigma})$.

11. It is easy to calculate $\bar{\phi}$ and $\bar{f}$ in terms of the fundamental concomitants of ϕ and f since the two covariants p and q have been used to give a typical representation of ϕ and f. We start from the formulæ to be found in GORDAN's *Invariantentheorie*, vol. 2, p. 327:

$$\bar{f}F = \tfrac{1}{2} A_{ff} p^2 + q^2,$$

$$-F^3\bar{\phi} = \left(F(pq) - \tfrac{1}{2}A_{ff}(sq)\right)p^3 - \tfrac{3}{2}MA_{ff}p^2q + \tfrac{3}{2}(A_{ff}L - A_{f\Delta}F)pq^2 + Mq^3,$$

where the meanings of M, A_{ff}, etc., are given in the preceding pages of GORDAN.

In calculating covariants, using the typical representation, we may proceed as if p and q were the variables x_1, x_2 provided we multiply the result by a power of (pq) equal to the weight of the covariant. For if I be a covariant of weight λ so that $I' = r^\lambda I$, r being the determinant of substitution, we may make the substitution $x_1' = p$, $x_2' = q$ of determinant $1/(pq)$; then $I' = [1/(pq)^\lambda]\ I$ or $I = (pq)^\lambda I'$ and I' is expressed in x_1', x_2' but $x_1' = p$, $x_2' = q$; hence the truth of the above statement.

Suppose then the cubic covariant $Q(\phi)$ of ϕ to be calculated from the typical representation; then the operation $D_{pq}\left(Q(\phi)\right)$ may be effected.

Let

$$Q(\phi) = C_0p^3 + C_1p^2q + C_2pq^2 + C_3q^3 = C_0(p - a_1q)(p - a_2q)(p - a_3q),$$

* CLEBSCH, *Binäre Formen.*

† CLEBSCH-LINDEMANN, *Geometrie*, p. 559.

where C_0, C_1, C_2 and C_3 are expressed in terms of the fundamental invariants of ϕ, f.

Then

$$D_{pq}\big(Q(\phi)\big) = (pq)^3 C_0 (p + 3a_1 q)(p + 3a_2 q)(p + 3a_3 q),$$

as a short computation shows, or

$$D_{pq}\big(Q(\phi)\big) = (pq)^3 (C_0 p^3 - 3C_1 p^2 q + 9C_2 pq^2 - 27(C_3 q^3)$$
$$= \phi' = c_0 p^3 + 3c_1 p^2 q + 3c_2 pq^2 + c_3 q^3 = c_x^3,$$

where

$$c_0 = (pq)^3 C_0, \quad -c_1 = (pq)^3 C_1, \quad c_2 = 3(pq)^3 C_2, \quad -c_3 = 27(pq)^3 C_3.$$

Next let f' be the Jacobian of (c_x^3, q), and pq; we find

$$f' = \tfrac{1}{2}(pq)^2 (c_0 p^2 - c_2 q^2) = d_x^2.$$

If we calculate $(cd)^2 c_x$ and $(cd)^2 (cd') d'_x$ which are of weights 2 and 3 we find

$$(cd)^2 c_x = \tfrac{1}{2}(pq)^4 (c_3 c_0 - c_1 c_2) q,$$
$$(cd)^2 (cd') d'_x = \tfrac{1}{4}(pq)^7 (c_0 c_1 c_2 - c_0^2 c_3) p,$$

so that

$$\sigma = \tfrac{1}{2}(pq)^4 (c_3 c_0 - c_1 c_2),$$
$$\rho = \tfrac{1}{4}(pq)^7 (c_0 c_1 c_2 - c_0^2 c_3),$$

and consequently $\rho\sigma^{-2}\phi'$ and $\sigma\rho^{-2} f''$ are expressed in terms of the form system of (Σ); that is, ϕ and f are so expressed, since

$$\rho\sigma^{-2}\phi' = \overline{\phi}, \quad \sigma\rho^{-1} f'' = \overline{f}.$$

THEOREM.—*The forms ϕ, f are expressed in terms of the fundamental concomitants of ϕ, f by the following equations:*

$$\phi' = c_0 p^3 + 3c_1 p^2 q + 3c_2 pq^2 + c_3 q^3,$$
$$c_0 = (pq)^3 C_0, \quad -c_1 = (pq)^3 C_1, \quad c_2 = 3(pq)^3 C_2, \quad -c_3 = 27(pq)^3 C_3,$$

where C_0, C_1, C_2, C_3 are defined by

$$Q(\phi) = C_0 p^3 + C_1 p^2 q + C_2 pq^2 + C_3 q^3;$$
$$f' = \tfrac{1}{2}(pq)^2 (c_0 p^2 - c_2 q^2),$$
$$\sigma = \tfrac{1}{2}(pq)^4 (c_3 c_0 - c_1 c_2),$$
$$\rho = \tfrac{1}{4}(pq)^7 (c_0 c_1 c_2 - c_0^2 c_3),$$

and

$$\phi = \frac{\rho}{\sigma^2}\phi', \quad f = \frac{\sigma}{\rho} f''.$$

IV.

The Involutions Belonging to ϕ, f and $\bar{\phi}, \bar{f}$.

12. In this section the involutions I_4 and $\bar{I}_4$ belonging to (Σ) and $(\bar{\Sigma})$ are determined.

Let

$$(pq, l)_1 = l_1, \qquad (pq, \bar{l})_1 = \bar{l}_1,$$
$$(pq, m)_1 = m_1, \qquad (pq, \bar{m})_1 = \bar{m}_1,$$
$$(pq, n)_1 = n_1; \qquad (pq, \bar{n})_1 = \bar{n}_1;$$

also

$$(p\bar{l}_1, ql_1)_1 = R_x^2 = R,$$
$$(p\bar{m}_1, qm_1)_1 = S_x^2 = S,$$
$$(p\bar{n}_1, qn_1)_1 = T_x^2 = T.$$

The three quadratics R, S, T are such that if we interchange the rôles of (Σ) and $(\bar{\Sigma})$ in forming them we find $-R, -S, -T$. Hence any relation between R, S, T and covariants of (Σ), $(\bar{\Sigma})$ is accompanied by a relation between $-R, -S, -T$ and the same covariants of $(\bar{\Sigma})$, (Σ). The three quadratics are branch quadratics of both I_4 and $\bar{I}_4$.

Theorem: *The forms Rl^2, Sm^2, Tn^2, f^2 belong to I_4 and the Jacobian of I_4 is proportional to ϕfq.*

From this result it follows that $R\bar{l}^2$, $S\bar{m}^2$, $T\bar{n}^2$, f^2 belong to $\bar{I}_4$ whose Jacobian is proportional to $\bar{\phi}fp$.

13. For the proof of this theorem some results on apolar systems of biquadratic forms are necessary.*

1°. To every involution $(\Pi) = \lambda f_1 + \mu f_2$ there is an apolar linear system $(\bar{\Pi}) = \lambda\phi_1 + \mu\phi_2 + \nu\phi_3$ such that every form of $(\bar{\Pi})$ is apolar to every form of (Π) and (Π) determines $(\bar{\Pi})$ and conversely.

2°. If a biquadratic form is apolar to the three forms ϕ_1, ϕ_2, ϕ_3 it must belong to (Π).

If, therefore, we can show that Rl^2, Sm^2, Tn^2, f^2 are all apolar to three biquadratic forms they must belong to the involution apolar to the three forms.

We are able to find three such forms, viz., $l^3(fl)_1$, $m^3(fl)_1$, $n^3(fn)_1$.

1°. It must be shown that $l^3(fl)_1$, $m^3(fm)_1$, $n^3(fn)_1$ are linearly independent, which is supposed of ϕ_1, ϕ_2, ϕ_3 above. Since $\phi = lmn$ we may regard l, m, n as arbitrary and also $(fl)_1$, $(fm)_1$ since f is so. For let a_x and b_x be any two linear forms, then if we take f to be the Jacobian of $a_x l_x$ and $b_x m_x$ we know that $(fl)_1$ and $(fm)_1$ are proportional to a_x and b_x, that is, we may choose f so that $(fl)_1$ and $(fm)_1$ are any two linear forms.

* Rosanes, Crelle's Journal, vol. 76.

Hence we may regard $l^3(fl)_1$ and $m^3(fm)_1$ as any two biquadratics having cubic factors.

If then a linear relation existed between $l^3(fl)_1$, $m^3(fm)_1$, $n^3(fn)_1$ it would take the form

$$n^3(fn)_1 = Al^3(fl)_1 + Bm^3(fm)_1$$

or $n^3(fn)_1$ would belong to the involution I_4 of $l^3(fl)_1$ and $m^3(fm)_1$. But a cube factor in a form of I_4 enters as a square in the Jacobian of I_4, hence the Jacobian of I_4 must be proportional to $l^2m^2n^2$. To see that this is not the case make a linear transformation by putting $l = x_1'$, $m = x_2'$; then denote $(fl)_1$ by $\rho(x_1' + 4\lambda x_2')$, $(fm)_1$ by $\sigma(4\mu x_1' + x_2')$. Then the Jacobian ϑ of $\rho x_1'^3(x_1' + 4\lambda x_2')$ and $\sigma x_2'^3(4\mu x_1' + x_2')$ must be a square. The computation gives

$$\vartheta = x_1'^2 x_2'^2\{3\mu x_1'^2 + (8\lambda\mu + 1)x_1'x_2' + 3\lambda x_2'^2\},$$

so that $3\mu x_1'^2 + (8\lambda\mu + 1)x_1'x_2' + 3\lambda x_2'^2$ must be a square, which it is not, since $(8\lambda\mu + 1)^2 - 36\lambda\mu \neq 0$ if λ and μ are arbitrary.

2°. To prove Rl^2, Sm^2, Tn^2, f^2 apolar to three forms involves twelve relations but it suffices to prove three of them; thus if $l^3(fl)_1$ is apolar to Sm^2 it must be to Tn^2, etc. It is sufficient to prove $l^3(fl)_1$ apolar to (*a*) Rl^2, (*b*) f^2, (*c*) Sm^2. The proof is in each case facilitated by the use of a proper normal form.

(*a*) $l^3(fl)_1$ is apolar to Rl^2 independently of the particular values of $(fl)_1$ or R. Make the substitution

$$x_1' = l, \quad x_2' = x_2,$$

then

$$l^3(fl)_1 = x_1'^3(a_1x_1' + a_2x_2'), \quad Rl^2 = x_1'^2(b_0x_1'^2 + 2b_1x_1'x_2' + b_2x_2'^2),$$

and these two forms are at once seen to be apolar. This involves the proof of three relations.

(*b*) To prove $l^3(fl)_1$ apolar to f^2b let $l = x_1'$, $(fl)_1 = x_2'$; then since $(fl)_1l$ is harmonic to f we must have $x_1'x_2'$ harmonic to f, that is $f = A_0x_1'^2 + A_2x_2'^2$ and $l^3(fl)_1 = x_1'^3x_2'$, but $x_1'^3x_2'$ is apolar to

$$(A_0x_1'^2 + A_2x_2'^2)^2 = A_0^2x_1'^4 + 2A_0A_2x_1'^2x_2'^2 + A_2^2x_2'^4.$$

This involves three relations.

(*c*) Since the relation $(l^3(fm)_1, Rm^2)_1 = 0$ is homogeneous in the coefficients of l, m, n we may in proving it use instead of l, m, n any forms proportional to them.

Let $x_1' = p$, $x_2' = q$, dropping the accents after transforming. Let ϕ be proportional to

$$x_1^3 + 3\lambda x_1^2x_2 + 3\mu x_1x_2^2 + \nu x_2^3 = a\phi = (x_1 + \xi_1x_2)(x_1 + \xi_2x_2)(x_1 + \xi_3x_2),$$

so that

$$\rho_1 l = (x_1 + \xi_1 x_2), \quad \rho_2 m = (x_1 + \xi_2 x_2), \quad \rho_3 n = (x_1 + \xi_3 x_2).$$

We know that f is harmonic to $x_1 x_2$, also to $(\phi p)_2$, that is, to

$$\lambda x_1^2 + \mu x_1 x_2 + \nu x_2^2;$$

so that f must be proportional to the Jacobian of $x_1 x_2$ and $(\phi p)_1$, or

$$\beta f = \lambda x_1^2 - \nu x_2^2,$$

where α and β are quantities which it is not necessary to determine.

If we put

$$\eta_i = \frac{3(\mu \xi_i - \nu)}{\lambda - \xi_i} \qquad (i = 1, 2, 3),$$

and use the method of forming $Q(\phi)$ described in section II we find $Q(\phi)$ proportional to

$$(3\xi_1 x_1 - \eta_1 x_2)(3\xi_2 x_1 - \eta_2 x_2)(3\xi_3 x_1 - \eta_3 x_2),$$

and performing the operation $D_{pq}\big(Q(\phi)\big)$ we find ϕ proportional to

$$(\xi_1 x_1 + \eta_1 x_2)(\xi_2 x_1 + \eta_2 x_2)(\xi_3 x_1 + \eta_3 x_2);$$

so that R is proportional to the Jacobian of $x_1(\xi_1 x_1 - \eta_1 x_2)$ and $x_2(x_1 - \xi_1 x_2)$, since $x_1 - \xi_1 x_2$, $\xi_1 x_1 - \eta_1 x_2$ are proportional to $l_1, \bar{l}_1$ or finally R, S, T are equal to

$$\sigma_1(x_1^2 - 2\xi_1 x_1 x_2 + \eta_1 x_2^2), \quad \sigma_2(x_1^2 - 2\xi_2 x_1 x_2 + \eta_2 x_2^2), \quad \sigma_3(x_1^2 - 2\xi_3 x_1 x_2 + \eta_3 x_2^2),$$

respectively.

Also $(fl)_1$ is proportional to

$$x_1 + \frac{\nu}{\xi_1 \lambda} x_2;$$

hence

$$(x_1 + \xi_1 x_2)^3 \left(x_1 + \frac{\nu}{\xi_1 \lambda} x_2\right)$$

must be apolar to

$$(x_1^2 - 2\xi_1 x_1 x_2 + \eta_2 x_2^2)(x_1 + \xi_2 x_2)^2.$$

To show this we note that if $a_x^4 = (xa)(x\beta)(x\gamma)(x\delta)$ is apolar to b_x^4 then $b_\alpha b_\beta b_\gamma b_\delta = 0$, so that if we take the third polar of Sm^2 with respect to $x_1 + \xi_1 x_2$ and the first polar of the result with respect to

$$x_1 + \frac{\nu}{\xi_1 \lambda} x_2,$$

the final result must be zero.

Doing this we obtain

$$\tfrac{1}{2}(\xi_2-\xi_1)\{(\xi_1^2+2\xi_1\xi_2+\eta_2)(\lambda\xi_1\xi_2-\nu) + (\xi_2-\xi_1)\big(\nu(\xi_1+\xi_2)+(\xi_1\xi_2+\eta_2)\lambda\xi_2\big)\}$$

which must be zero, or since $\xi_1 \neq \xi_2$,

$$(\xi_1^2+2\xi_1\xi_2+\eta_2)(\lambda\xi_1\xi_2-\nu)+(\xi_2-\xi_1)\big(\nu(\xi_1+\xi_2)+(\xi_1\xi_2+\eta_2)\lambda\xi_2\big)$$

must be zero; or solving for η_2 and remembering that

$$3\lambda=\xi_1+\xi_2+\xi_3,$$
$$3\mu=\xi_2\xi_3+\xi_3\xi_1+\xi_1\xi_2,$$
$$\nu=\xi_1\xi_2\xi_3,$$

η_2 must equal

$$\frac{(\xi_1^2+2\xi_1\xi_2)(\xi_2\lambda-\xi_2\xi_3)-(\xi_1-\xi_2)\big(\lambda\xi_1\xi_2+\xi_1\xi_2(\xi_1+\xi_2)\big)}{\xi_2\xi_3-\lambda(2\xi_2-\xi_1)}.$$

The denominator $\xi_2\xi_3-\lambda(2\xi_2-\xi_1)$ is equal to

$$(\lambda-\xi_2)(\xi_1+\xi_2),$$

and the numerator is equal to

$$(\xi_1+\xi_2)\{(\xi_2-\xi_1)\xi_2\xi_3+\xi_1(\xi_2\lambda-\xi_2\xi_3)\}+\xi_1\xi_2(\xi_2\lambda-\xi_2\xi_3)-(\xi_1-\xi_2)\lambda\xi_1\xi_2,$$

and

$$\xi_1\xi_2(\xi_2\lambda-\xi_2\xi_3)-(\xi_1-\xi_2)\lambda\xi_1\xi_2=-(\lambda\xi_2\xi_1^2+\xi_2\nu-2\lambda\xi_2^2\xi_1)$$
$$=-\{\lambda\xi_1\xi_2(\xi_1+\xi_2)-\xi_1^2\xi_2^2-\xi_2^3\xi_1\}=(\xi_1+\xi_2)(\xi_2^2\xi_1-\lambda\xi_1\xi_2);$$

or the numerator of η_2 is equal to

$$(\xi_1+\xi_2)(\xi_2^2\xi_3-\xi_1\xi_2\xi_3+\lambda\xi_1\xi_2-\xi_1\xi_2\xi_3-\lambda\xi_1\xi_2+\xi_2^2\xi_1)$$
$$=(\xi_1+\xi_2)\{\xi_2(\xi_2\xi_3+\xi_3\xi_1+\xi_1\xi_2)-3\xi_1\xi_2\xi_3\}=3(\xi_1+\xi_2)(\mu\xi_2-\nu)$$

or finally, η_2 must equal

$$\frac{3(\xi_1+\xi_2)(\mu\xi_2-\nu)}{(\lambda-\xi_1)(\lambda-\xi_2)}=\frac{3(\mu\xi_2-\nu)}{\lambda-\xi_2},$$

which is the definition of η_2. Hence the relation from which we started must be true. This proves (c) involving six relations and the theorem is proved in all its parts if we show that the Jacobian of I_4 is proportional to $\phi f q$. This follows immediately from section I where it was proved that the Jacobian of I_4 determined from ϕ and f is proportional to $\phi f'(x\delta)$, where $(x\delta)=(ab)^2(ab')b_x'$ which is the definition of q.

In virtue of the reasoning at the end of section III it follows that Rl^2, Sm^2, Tn^2, f^2 belong to I_4 whose Jacobian is proportional to $\bar{\phi}\bar{f}\bar{q}$, that is to $\phi f p$.

It follows that I_4 and $\bar{I}_4$ determined by (Σ) and $(\bar{\Sigma})$ have three branch quadratics in common and the two covariants p, q are the fourth double elements, q in I_4 and p in $\bar{I}_4$.

I_4 derived from (Σ)	$\bar{I}_4$ derived from $(\bar{\Sigma})$
Rl^2, Sm^2, Tn^2, f^2	$R\bar{l}^2$, $S\bar{m}^2$, $T\bar{n}^2$, $\bar{f}^2$
with double elements given by	with double elements given by
$\phi f q$.	$\bar{\phi}\bar{f}p$.

14. In section III ϕ and f were expressed in terms of the fundamental concomitants of ϕ and f by means of the typical representation by p and q. Since f^2 is a form of I_4 if we express one other form of I_4 in terms of the concomitants of ϕ, f we have every form of I_4 so expressed. The branch form whose double element is q will be rational in the concomitants of ϕ, f. Let it be equal to $F = q^2(ap^2 + 2\gamma pq + \beta q^2)$ and let f be $Ap^2 + Bq^2$ (there is no term in pq in the typical representation of f). Then the Jacobian of F and f^2 must be proportional to $\phi f q$, or the Jacobian of F, f to ϕq, or finally we may take α, β, γ so that

$$q \begin{vmatrix} Ap, & Bq \\ q(ap + \gamma q), & q(\gamma p + \beta q) + 2(ap^2 + \beta q^2 + \gamma pq) \end{vmatrix} = \phi q,$$

or

$$2A\alpha p^3 + 3A\gamma p^2 q + (3A\beta - \alpha B)pq^2 - B\gamma q^3 = \phi,$$

and since ϕ is expressed in terms of p and q, by equating coefficients α, β, γ may be found and consequently F expressed as concomitant to ϕ, f.

If we express the form of I_4, which contains p^2, in terms of p, q, we can express every form of I_4 in terms of p, q, for $\bar{f}^2$ is already so expressed.

V.

The reducible Hyperelliptic Integrals.

15. Since I_4 and $\bar{I}_4$ have three branch quadratics in common and q and p are the fourth double elements, we have from the Introduction the

Theorem: *The integrals*

$$\int \frac{q(x\,dx)}{\sqrt{RST}}, \qquad \int \frac{p(x\,dx)}{\sqrt{RST}}$$

are reducible to elliptic integrals by transformations of the fourth order I_4, $\bar{I}_4$.

Having given one reducible integral and its reducing transformation we can solve the problem to find the second one belonging to the same irrationality and its reducing transformation. For the reducing transformation being known the form ϕ is known and also f; from ϕ and f we deduce the covariant $(ab)^2a_x$ which is the numerator of the second reducible integral; the second reducing transformation being given by the involution $\bar{I}_4$ derived from $\bar{\phi}$ and f, which can be found from ϕ and f.

We can enunciate the following theorem:

THEOREM: *If $\phi = a_x^3$ denotes the product of the three double elements complementary to the three quadratic factors of the sextic of a reducible integral, and $f = b_x^2$ denotes the form whose square occurs among the forms of the reducing involution, then the numerator of the second reducible integral belonging to the same irrationality is equal to $(ab)^2a_x$.*

16. It is possible to give another definition of the numerator of the second integral which resembles that given by Professor BOLZA* for the transformation of order 3 and also that for order 2. For we have shown that if l_x denotes a double element of an I_4 containing a complete square f^2, then $l^3(fl)_1$ is apolar to the forms of I_4; but q is a double element of I_4 and $(fq)_1$ is proportional to p, hence q^3p is apolar to I_4. We can enunciate the theorem:

THEOREM: *If $(x\delta)$ denotes the numerator of the first reducible integral and $(x\delta)$ of the second and I_4 the reducing involution, then $(x\delta)^3(x\delta)$ is apolar to the forms of I_4 and this is sufficient to determine $(x\delta)$.*

The following conjectural theorem suggests itself: *If $(x\delta)$ and $(x\delta)$ denote as before the numerators of the integrals—reducible by a transformation of order k—and I_k the reducing involution, then $(x\delta)^{k-1}(x\delta)$ is apolar to the forms of I_k.* This surely holds for $k = 2$, 3 and, as has been shown in this paper, for $k = 4$.

VI.

NORMAL FORM OF REDUCIBLE INTEGRALS.

17. The normal form used by Professor BOLZA† is found by introducing p and q as variables, that is, by introducing the numerators of the two integrals. This has been done for a special purpose in section IV, and the formulæ there developed have only to be supplemented; we found

$$R = \sigma_1(x_1^2 - 2\xi_1x_1x_2 + \eta_1x_2^2),$$
$$S = \sigma_2(x_1^2 - 2\xi_2x_1x_2 + \eta_2x_2^2),$$
$$T = \sigma_3(x_1^2 - 2\xi_3x_1x_2 + \eta_3x_2^2),$$

*Mathematische Annalen, vol. 50, p. 314.

†Dissertation, Mathematische Annalen, vol. 28.

$$RST = \sigma_1\sigma_2\sigma_3\{\nu' x_1^6 - 6\lambda\nu' x_1^5 x_2 + 3(4\mu\nu' + \lambda\mu')x_1^4 x_2^2$$
$$+ 2(\lambda\lambda' + 5\nu\nu')x_1^3 x_2^3 + 3(4\mu'\nu' + \lambda'\mu)x_1^2 x_2^4 - 6\lambda'\nu x_1 x_2^5 + \nu x_2^6\},$$

where

$$\lambda' = -\tfrac{1}{3}\frac{2\lambda^2\nu - \lambda\mu^2 - \mu\nu}{-\nu^2 + 3\lambda\mu\nu - 2\mu^3},$$

$$\mu' = \tfrac{1}{9}\frac{\lambda^2\mu + \lambda\nu - 2\mu^2}{-\nu^2 + 3\lambda\mu\nu - 2\mu^3},$$

$$\nu' = -\tfrac{1}{27}\frac{2\lambda^3 - 3\lambda\mu + \nu}{-\nu^2 + 3\lambda\mu\nu - 2\mu^3}.$$

Professor BOLZA used implicitly a form corresponding to ϕ, viz.:

$$x_2^3 + 3\lambda' x_2^2 x_1 + 3\mu' x_2 x_1^2 + \nu' x_1^3,$$

and made the remark that if the cubic covariant of

$$x_1^3 + 3\lambda x_1^2 x_2 + 3\mu x_1 x_2^2 + \nu x_2^3$$

is

$$C_0 x_1^3 + 3C_1 x_1^2 x_2 + 3C_2 x_1 x_2^2 + C_3 x_2^3,$$

then

$$\lambda' = -\tfrac{1}{3}\frac{C_2}{C_3}, \quad \mu' = \tfrac{1}{9}\frac{C_1}{C_3}, \quad \nu' = -\tfrac{1}{27}\frac{C_0}{C_3},$$

which was the starting point for discovering the relations in sections III and IV.

The reducible integrals are

$$\int \frac{x_1(x\,dx)}{\sqrt{R}}, \quad \int \frac{x_2(x\,dx)}{\sqrt{R}},$$

where $R = RST$ is expressed in x_1, x_2 above.

The reducing substitution for the first integral is given by any two of the forms

$$\sigma_i(x_1^2 - 2\xi_i x_1 x_2 + \eta_i x_2^2)(x_1 + \xi_i x_2)^2 \qquad (i = 1, 2, 3).$$

The form f is equal to $(\lambda x_1^2 - \nu x_2^2)$.

VII.

MISCELLANEOUS RESULTS ON BIQUADRATIC INVOLUTIONS CONTAINING A COMPLETE SQUARE.

18. In section VII are collected some miscellaneous results relating to special biquadratic involutions containing a square and their geometrical representations.

(*A*) It is a known theorem that the Jacobians of two apolar systems are the same.* Since the roots of the Jacobian are multiple roots in forms of the system from which it is derived this theorem states that if (xa) is a double element of a biquadratic involution (II) it is a triple element of the apolar system $(\overline{\text{II}})$; hence $(xa)^3(x\bar{a})$ is a form of $(\overline{\text{II}})$. When (II) is a special involution we are able to determine $(x\bar{a})$ from section IV for it was there shown that $(xa) = \left(f, (xa)\right)_1$, f being the form whose square belongs to (II).

THEOREM.—*If* (xa) *is a double element of a special biquadratic involution* I_4 *containing a square* f^2, *then* $(xa)^3\left(f, (xa)\right)_1$ *belongs to the system apolar to* I_4.

(*B*) The theorem of section I which says that the form there denoted by θ is apolar to the involution $\lambda u + \mu f^2$ admits of a geometrical interpretation. Let a_x^4, β_x^4, γ_x^4 be three forms apolar to u_x^4 and v_x^4. The rational quartic C_4† given by the parameter representation

$$\rho x_1 = a_x^4,$$

$$\rho x_2 = \beta_x^4,$$

$$\rho x_3 = \gamma_x^4,$$

is intersected by a straight line $\lambda x_1 + \mu x_2 + \nu x_3$ in 4 points whose parameters satisfy the equation:

$$\lambda a_x^4 + \mu \beta_x^4 + \nu \gamma_x^4 = 0,$$

that is, they are given by a form apolar to u_x^4 and v_x^4.

If $\lambda x_1 + \mu x_2 + \nu x_3$ is an inflexional tangent of C_4 the equation

$$\lambda a_x^4 + \mu \beta_x^4 + \nu \gamma_x^4 = 0$$

has a triple root which is the parameter value for the point of inflexion. By the theorem already quoted in this section this must be a root of the Jacobian of u_x^4 and v_x^4. When this is applied to the special involution, $\theta = 0$ must give 4 points of inflexion; but θ is a form of the system apolar to the involution, that is $\theta = 0$ gives 4 collinear points, hence the

THEOREM: *Four of the points of inflexion of a special* C_4 *are collinear.*

(*C*) Another definition can be given of the linear factors of ϕ, which is adapted to their determination when a twisted cubic C_3 is used to carry the binary variables.‡ Using the notation of section II, we have the

THEOREM: Lq^2 *is apolar to* $(pq, l)_1 p^2$.

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*STEPHANOS: *Sur les faisceaux ayant une même jacobienne*, Mémoires par divers savants, vol. 27.

† MEYER: *Apolarität und Rationale Curven.*

‡ STURM, Crelle's Journal, vol. 86.

Proof.—Referring to the definition, l is proportional to $p(qL) + 3q(pL)$, hence $(pq, \bar{l})_1$ is proportional to $p(pq)(qL) - 3q(pq)(pL)$.

To express L in a convenient form we notice that $\big(pq, (pq, L)_1\big)_1$ is proportional to L, for we have taken the harmonic $(pq, L)_1$ of L with regard to pq and then of the linear form so found with regard to pq again, which must give the original L up to a factor.

Hence,

$$(pq, L)_1 = \tfrac{1}{2}\big(p(qL) + q(pL)\big),$$

and therefore

$$\big(pq, (pq, L)_1\big)_1 = \tfrac{1}{4}\{-p(pq)(qL) + q(pq)(pL)\},$$

therefore

$$\{-p(pq)(qL) + q(pq)(pL)\}q^2$$

must be apolar to

$$\{(pq)(qL) - 3q(pq)(pL)\}p^2;$$

that is, if we put $x_1' = p$, $x_2' = q$,

$$-x_1'x_2'^2(pq)(qL) + x_2'^3(pq)(pL)$$

must be apolar to

$$x_1'^3(pq)(qL) - 3x_1'^2x_2'(pq)(pL),$$

that is,

$$(pq)^2(qL)(pL) - (pq)^2(qL)(pL)$$

must be zero. This is true and therefore the theorem is proved.

The apolarity of cubic forms admits of geometrical representation (STURM, loc. cit.). Let a_x^3 be a cubic, to it will correspond a plane having a null point; if b_x^3 is another cubic apolar to a_x^3 then the plane corresponding to b_x^3 will pass through the null point of the plane corresponding to a_x^3, and conversely; if the plane passes through that null point b_x^3 will be apolar to a_x^3.

There is also a geometrical representation of the cubic covariant Q of $\phi = a_x^3$ on C_3. Call A the plane corresponding to ϕ, then A contains one line which is the intersection of two osculating planes of C_3; call these B and C. Then the plane D, which in the pencil containing A, B, C divides A harmonically from B and C, meets C_3 in 3 points given by $Q(\phi) = 0$.

Suppose we start with ϕ and f, then we can construct p and q (STURM, loc. cit.); also $Q(\phi) = LMN$. Then by the theorem last proved we can find the point $(pq, \bar{l})_1 = 0$, viz: the plane containing the tangent to C_3 at $p = 0$ and the null point of the plane through the point $L = 0$ and the tangent at $q = 0$ must meet C_3 in three points among which is the point $p = 0$ counted twice and the third of which is the point $(pq, l)_1 = 0$.

If we take the harmonic of the point $(pq, \bar{l})_1 = 0$ with respect to $pq = 0$ we reach the point $\bar{l} = 0$ and thus the points $\phi = 0$ may be found.

THEOREM: *Given the plane corresponding to* $\phi = 0$ *and the line corresponding to* $f = 0$, *it is possible to find on the twisted cubic* C_3 *the plane corresponding to* $\phi = 0$.

The harmonic of a point with respect to two others may be found (MEYER, loc. cit.) by considering the hyperboloids through C_3. But if we wish the harmonic of a with respect to b and c it will be found among the points given by the cubic covariant of a, b, c, viz: it is that one which is separated from a by b and c in virtue of the definition of cubic covariant used in section II.

This is a simpler way of reaching it because of the simple geometrical construction of the cubic covariant.

ON THE THEORY OF IMPROPER DEFINITE INTEGRALS*

BY

ELIAKIM HASTINGS MOORE

§ 1.

INTRODUCTION. THE FOUR CURRENT TYPES OF IMPROPER INTEGRALS.

1°. In this paper I wish to define a system of types of improper simple definite integrals, a system embracing in particular the four current types; of the theory of the general type I give at present merely the elements, the methods employed, however, being characteristic.

The four current types are compared in § 1 2°–13°. By way of generalization of their diversities the new types arise (11°–17°). As the desirable basis for the new types I propose (16°) an extension of the notion of the proper simple definite integral; this involves likewise an extension of the notions of the four current types.

In § 2 I state in convenient notations a body of elementary properties of the general type of integrals. These properties with two definitional processes of induction developed in §§ 3, 4 serve as the basis for the definition in § 5 of the system of types of improper integrals related to the (extended) type of proper integrals defined in 16°.

2°. If the definite integral

$$\int_a^b F(x)\,dx \tag{1}$$

is a proper integral, the interval ab of integration is finite and the integrand function $F(x)$ exists (everywhere defined but not necessarily in a single valued way) on ab as a limited function.

3°. IMPROPER INTEGRALS OVER FINITE INTERVALS.—We call to mind the four current types of improper integrals (1) over finite intervals ab. The following presuppositions are common to the four types. On ab the function $F(x)$ has a singular point-set Z. In every neighborhood of a point ζ of Z the function $F(x)$ is unlimited. On every interval of ab containing no point ζ the function $F(x)$ is properly integrable.

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4°. TYPE I.—In the first type the set Z is reducible, that is, for some (lowest) ordinal * a of CANTOR'S first or second class of ordinals, the derived set $Z^{(a)} \equiv 0$; hence the set Z has the content zero, and it is finite or numerably infinite; and conversely, any set finite or numerably infinite is reducible. CAUCHY and RIEMANN considered the case of finite sets $Z(a = 1)$, and DU BOIS-REYMOND (1875: Crelle's Journal, vol. 79, pp. 36, 45), and DINI (1878: DINI-LÜROTH, pp. 404–445) the case of sets Z of the first kind † $(a = \nu = 1, 2, 3, \cdots)$, and quite recently SCHOENFLIES (1900: *Bericht*, pp. 185–186, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 8, no. 2) the general case $(a = a)$.

5°. TYPE II.—In the second type the set Z is any set of content zero. For a systematic exposition of the theory of these the (general) HARNACK integrals of 1883–1884 I refer to my paper in the last number of this journal (pp. 296–330); in the introduction are indicated the relations with the equivalent JORDAN theory of 1894.

The theories of these two types have this fundamental difference: the integral of the second type is defined by a definition covering all cases at once, while all cases of the first type are covered in that the integral for the case $a = 1$ is defined directly and then two definitional processes are given which serve to define, in the first place, the integral for the case $a = a_0 + 1$ in terms of integrals for the case $a = a_0$ and, in the second place, the integral for the case $a = a_\omega$ in terms of a numerably infinite set of integrals for the cases $a = a_\nu (\nu = 1, 2, 3, \cdots)$, where the ordinal a_ω is next higher than the set $\{a_\nu\}$. Thus the definition of a particular integral of the second type involves a single limiting process, and that of one of the first type involves a finite or a numerably infinite system of limiting processes.

6°. In the comparison of two types of integrals, presupposing that in each theory every integral (1) when existent has a definite finite value, we say for brevity that the two types are *accordant* or *in accord* in case every integral existent in each theory has in the two theories equal values; and of two accordant types we say that the first *contains* or *includes* the second if every integral existent in the second theory exists also in the first. Two types are *equivalent* if each includes the other.

7°. Now to compare the first two theories, we see that the second theory has wider application than the first theory, since the second theory covers the cases of irreducible sets Z. As to integrals with reducible sets Z, the two types are

* That is ordinal number (*Ordinalzahl*).

† *Gattung*.

well known to be equivalent for the case $a = 1$. For the cases* $a > 1$, however, while it is readily seen that the first type contains the second, the two types are not equivalent; for example ($a = 2$), the integral (31) p. 330 of my paper cited above exists or not according as it is of the first or second type.

8°. TYPE III.—The third type is that of HÖLDER (Mathematische Annalen, vol. 24 (1884), p. 90 ff.); the set Z is reducible. HÖLDER defines primarily an indefinite integral

$$\int F(x)\,dx \tag{2}$$

on the x-interval ab, viz., as any function $\phi(x)$ continuous on ab and such that for any two points x' x'' of ab, whose interval $x'x''$ contains no point ζ one has the relation

$$\phi(x'') - \phi(x') = \int_{x'}^{x''} F(x)\,dx. \tag{3}$$

This definition would apply to any set Z. The restriction to reducible sets Z is made in order that two such indefinite integrals on ab may differ on ab by a constant, and thus that the definite integral:

$$\int_{a'}^{b'} F(x)\,dx = \phi(b') - \phi(a'), \tag{4}$$

where $a'b'$ are any two points of ab, may be independent of the indefinite integral $\phi(x)$ used in its definition.

The first and the third types of definite integrals are easily seen to be equivalent. Indeed it is possible that HÖLDER had in mind the final extension, later made by SCHOENFLIES, of the integrals of the first type to the case of any ordinal a.

9°. TYPE IV.—The fourth type is that of DE LA VALLÉE-POUSSIN (1892: Liouville's Journal, ser. 4, vol. 8, pp. 421–467). For the general set Z of content † zero a primary definition is given by a single limiting process which is of such a nature that the integral (1) if existent is absolutely existent. Then, on

*I have found only the two following notes of comparison of the two types for these cases:

STOLZ (Wiener Berichte, vol. 107_{2a} (1898), pp. 207–224: cf. §2) states that for the case $a = \nu > 1$ the second type is not evidently ("nicht ohne Weiteres") to be identified with the first.

SCHOENFLIES (loc. cit., p. 188, ll. 21–24) discriminates in general the cases of sets Z reducible and irreducible by introducing the terms: integrals of the first and second kinds (*Arten*), and thus distinctly implies that for reducible sets Z the two types are equivalent.—The proper discrimination is by the type.

†SCHOENFLIES (loc. cit., pp. 178, 187, §5) points out and emphasizes the fact that this condition if not inserted in the definition may be deduced as a conclusion, in any case in which the (modified) definition yields an integral.

the basis of the proper and of these absolutely existent improper integrals, for the general reducible set Z_0 of all points ζ of Z in whose neighborhood the function $F(x)$ is not absolutely integrable the non-absolutely existent integrals are defined * as in the first theory.

It is to be noted that SCHOENFLIES (loc. cit., pp. 188, 189) gives vague indications that the case of irreducible sets Z_0 may be included by suitable phrasing of the secondary definition, and indeed that the primary definition may be so modified as to include the non-absolutely existent integrals.

The second and the fourth types are equivalent in so far as absolutely existent integrals are concerned,† and, likewise, as to non-absolutely existent integrals, in the case $a = 1$ for the reducible set Z_0, while in the cases $a > 1$ the fourth type contains but is not equivalent to the second type; the example (for $a = 2$) cited above (7°) serves here also.

10°. To summarize: The HARNACK (second) type of integral (1) has the widest range of application and it involves a single limiting process. The other types involve in general a system of limiting processes. Every type contains the corresponding subtype of the HARNACK type. Every subtype involving only one limiting process is equivalent to the corresponding HARNACK subtype.

In every theory the integral symbol (1) has a unique definition.

11°. For a given singular set Z the system of types of improper integrals (1) to be defined in this paper arises from the four current types by way of generalization of their diversities. The reader will doubtless have foreseen the character of the general type to be proposed. It bears to the HARNACK type a relation which generalizes (not uniquely) the relation borne by the first type ($a = a$) to the CAUCHY subtype ($a = 1$). The integral symbol (1) for any type of the system receives a unique definition depending on the type, or, otherwise expressed, on the way in which the set Z is utilized in the limiting process or processes involved in the definition.

At present in connection with the definition of the system of types I consider briefly the theory of the general type of the system but not the relations amongst the various types. I conjecture however that this relation is (as amongst the current types) simply that every type is in accord with every other type. If this relation is indeed valid, we may obviously unite or fuse all the types of the system into a new single type in which the symbol (1) has again a unique definition.‡

* The definition given by DE LA VALLÉE-POUSSIN (loc. cit., p. 453, § 55) is restricted to the case of reducible sets Z_0 of the first kind.

† Cf. STOLZ (loc. cit., introduction).

‡ If in any particular type of the system a symbol (1) has a value, it shall have that value in the new type. If in every type of the system the symbol (1) has no value, it shall have no value in the new type.

12°. Since in the theory of a particular type of integrals the definite integral function

$$J(x) = \int_a^x F(x)\,dx \qquad (x \text{ on } ab) \tag{5}$$

of a function $F(x)$, properly or improperly integrable from a to b, is a continuous function of x on ab, and quâ $\phi(x)$ satisfies the relation (3), we have in that theory, even for irreducible sets Z, means of selecting from amongst all HÖLDER'S functions $\phi(x)$ a system every two of which differ on ab by a constant, viz., the system

$$\phi(x) = J(x) + C, \tag{6}$$

where C is an arbitrary constant. For this theory these functions $\phi(x)$ may then properly receive the indefinite integral notation:

$$\phi(x) = \int F(x)\,dx. \tag{7}$$

In case we may fuse all the types of the system into a new type, as suggested in 11°, we obtain in the new theory a unique system of indefinite integrals (7).

13°. IMPROPER INTEGRALS OVER INFINITE INTERVALS.—In each of the four current types the improper integral (1) over an infinite interval ab is defined either explicitly or presumptively as the limit of the proper or improper integral over the finite subinterval $a'b'$ of ab as $a'b'$ converges to ab.* Thus for each type the theory of the improper integrals over infinite intervals depends upon that of the improper integrals over finite intervals.

It is, however, possible to define the former integrals as transforms of the latter integrals. The transformation

$$x = \psi(\bar{x}) = -(\bar{x} \pm 1)^{-1} \pm 1 \qquad (x \text{ on the interval } {}^{-\infty \cdots 0}_{0 \cdots \infty}) \tag{8}$$

throws the complete† x-line $(-\infty \leqq x \leqq \infty)$ in a one-to-one way on the $\bar{x}$-interval $-1 \leqq \bar{x} \leqq +1$; as x traverses the x-line in the positive sense x traverses the $\bar{x}$-interval in the same sense; $\psi(\bar{x})$ is a continuous function of $\bar{x}$ with a continuous positive derivative $\psi'(\bar{x}) = (\bar{x} \pm 1)^{-2}$ on the $\bar{x}$-interval excluding its extremities ∓ 1. Then, setting

* We define in like manner the content $\gamma(\Xi)$ of a point-set Ξ lying on an infinite interval ab as the limit of the content $\gamma(\Xi')$ of the subset Ξ' of Ξ lying on the finite subinterval $a'b'$ of ab as $a'b'$ converges to ab.

† It is here permissible and convenient to admit the definite infinities ∞, $-\infty$ as arguments and functional values. For a function $f(x)$ the two numbers

$$f(\infty), \quad f(\infty - 0) \equiv \mathop{\mathrm{L}}_{x=\infty} f(x)$$

are definitionally distinct, and so are $f(-\infty), f(-\infty + 0)$. The function $f(x)$ is continuous at ∞ if $f(\infty), f(\infty - 0)$ exist and are equal.

(9) $F(\bar{x}) \equiv F\big(\psi(\bar{x})\big)\,\psi'(\bar{x}) = F\big(\psi(\bar{x})\big)\,(\bar{x} \pm 1)^{-2}$ (x on the interval $\begin{smallmatrix}-\infty & \cdots & 0\\ 0 & \cdots & \infty\end{smallmatrix}$),

and denoting by a, $\bar{a}$ and b, $\bar{b}$ two pairs of corresponding values of x, $\bar{x}$ respectively, we have the relation of transformation:

$$\int_a^b F(x)\,dx = \int_{\bar{a}}^{\bar{b}} F(\bar{x})\,d\bar{x}. \tag{10}$$

This relation is valid for finite intervals ab.—We may use the relation to define the integrals over infinite intervals, viz., as transforms of those over finite intervals.

Similarly, for any type of integrals, we may define the integrals over infinite intervals either as limits of integrals of the same type or as transforms of integrals of a transformed type, these integrals being in each case over finite intervals.

It seems to be desirable to adopt the former and usual definition as the definitive definition for use in the applications of the theory of any type of integrals, but in the development of the theory to adopt the latter new definition. For it is evident that an integral under the new definition is one under the old definition, while an integral under the old definition, if not one under the new definition, is one under the new definition for a closely related type of integrals; and so the theory is richer by the adoption of the old definition while its development (if made for the general type of integrals) is somewhat simpler by the adoption of the new definition.

For the first, third, and fourth types the two definitions are equivalent. In the second type, however, it is possible to have, for example, a HARNACK integral $\int_1^\infty F(x)\,dx$ (according to the old definition) whose transformed HARNACK integral $\int_{\frac{1}{2}}^1 \overline{F}(\bar{x})\,d\bar{x}$ does not exist: the example of §3 9° of my paper already cited is to the point,—on the x-interval $\frac{1}{2}1 = ab$ the present function $F(x)$ being the function $F(x) + G(x)$ of the reference.

14°. For the sake of clearness of thought and also of greater generality we replace the point-set Z of singularities ζ of the integrand function $F(x)$ by any point-set Ξ of content zero.* The general type of integral is then in a certain sense a function or an aspect of Ξ. The integral (1) depends upon the type, the interval ab, and the integrand function $F(x)$. If the symbol (1) has value, the function $F(x)$ has certain relations to Ξ, e. g., on every interval of ab containing no point ξ of Ξ the function $F(x)$ is properly integrable; and further, the limiting relations implied in the type hold.

15°. For the purpose of the later theory the content $\gamma(\Xi)$ of Ξ needs to be zero. But the definitions and theory of the present paper are effective also for

* But cf. 15°.

sets Ξ of arbitrary content $\gamma(\Xi)$; since this fact has interest, the more general form of presentation is adopted.

16°. THE MORE GENERAL PROPER DEFINITE INTEGRALS.—I extend the notion of the proper definite integral (1) as follows:

The interval ab is a finite interval. The integrand function $F(x)$ exists on ab, or, at least, on ab apart from a point-set Ω of content zero, as a function everywhere defined* and limited;† and a function $G(x)$, arising from $F(x)$ by modification of its determination on Ω so that $G(x)$ is on ab everywhere defined and limited, is properly integrable from a to b in the usual sense. Then we set in general

$$\int_a^b F(x)\,dx = \int_a^b G(x)\,dx. \tag{11}$$

In case on the set Ω $F(x)$ itself is everywhere defined and limited, the integral (1) exists as a proper integral in the usual sense, and the equality (11) is well known.

In other cases the equality (11) serves to define the integral (1) as a definite finite number, a proper definite integral in the new sense.

By virtue of this definition of the type of proper integrals every existing definite integral (1) of this type or of any derivative type defined in this paper (§5) retains its type and value under any modification of the determination of the integrand function on a point-set of content zero. This statement is embodied in theorem II of §2; and it is to be compared with the remark of §5 7° (p. 329) of my paper cited above (5°).

§2.

NOTATIONS AND ELEMENTARY PROPERTIES OF THE GENERAL TYPE OF INTEGRALS.

1°. With respect to any linear point-set Ξ (§1 15°) we are to consider a system of types of improper definite integrals (§1 (1)), and especially the general (any particular) type of the system. For this type we use the notation $\mathbf{I}$ ("integral"), instead of the integral sign $\int$ with an affix indicating the type in question. Thus, the symbol:

$$\mathbf{I}_a^b F(x)\,dx, \tag{1}$$

where a and b are definite points (finite or infinite) and where $F(x)$ is a definite function of x on the interval ab, denotes a definite integral of the type $\mathbf{I}$,

* Not necessarily in a single valued way.

† On the set Ω $F(x)$ need not be defined, and, so far as it is defined, it need not be limited.

which is either existent and of definite finite value or non-existent. We speak of the $\mathbf{I}$-integral (1) and of functions $F(x)$ $\mathbf{I}$-integrable from a to b.

The closed set $\Xi + \Xi'$ is supposed to contain the two infinite points $(\pm \infty)$. If $\Xi = \{+\infty, -\infty\}$ the (only) type $\mathbf{I}$ of the set Ξ is the type of the proper definite integrals of §1 16° over finite intervals and

The integrals of type $\mathbf{I}$ over subintervals of a fixed interval ab constitute a type $\mathbf{I}_{ab}$. With respect to the type $\mathbf{I}_{ab}$ all sets Ξ having the same subset Ξ_{ab} lying on ab are equivalent.

2°. The following theorems (I–VIII) of the theory of the proper definite integrals defined in §1 16° hold also in the theory of the integrals of the type $\mathbf{I}$.

I*. *The function* 0 *is* $\mathbf{I}$*-integrable from* a *to* b, *and* $\mathbf{I}_a^b 0\,dx = 0$.

II. *A change in the determination of the integrand function* $F(x)$ *in the points of a point-set of content* † *zero makes no change in the determination of the* $\mathbf{I}$*-integral* $\mathbf{I}_a^b F(x)\,dx$.

FIXED HYPOTHESIS.—*The function* $F(x)$ *in* III–VIII (IV′ *apart) is supposed to be* $\mathbf{I}$*-integrable from* a *to* b.

III. *The function* $F(x)$ *is likewise* $\mathbf{I}$*-integrable from* x_1 *to* x_2, *where* $x_1 x_2$ *are any two points of* ab.

IV. *For any three points* $x_1 x_2 x_3$ *of the interval* ab

$$\mathbf{I}_{x_1}^{x_2} F(x)\,dx + \mathbf{I}_{x_2}^{x_3} F(x)\,dx = \mathbf{I}_{x_1}^{x_3} F(x)\,dx;$$

and accordingly

$$\mathbf{I}_{x_1}^{x_1} F(x)\,dx = 0, \quad \mathbf{I}_{x_2}^{x_1} F(x)\,dx = -\mathbf{I}_{x_1}^{x_2} F(x)\,dx.$$

IV′. *If* $F(x)$ *is* $\mathbf{I}$*-integrable from* x_1 *to* x_2 *and from* x_2 *to* x_3, *then it is* $\mathbf{I}$*-integrable from* x_1 *to* x_3, *and the equality of* IV *holds.*

V. *For every positive number* ϵ *there exists a positive number* δ_ϵ *such that* $|\mathbf{I}_{x_1}^{x_2} F(x)\,dx| < \epsilon$ *for every two points* $x_1 x_2$ *of* ab *for which* $|x_1 - x_2| < \delta_\epsilon$.

In the light of theorem III, theorem V is the affirmation of the uniform continuity on ab of the definite integral function

$$J(X) = \mathbf{I}_a^X F(x)\,dx \qquad (X \text{ on } ab), \tag{2}$$

an affirmation known to include and to follow‡ from the statement:

V′. *The definite integral function* $J(X)$ (2) *is a continuous function of the variable upper limit* X *on the* X*-interval* ab.

VI. *The function* $c\,F(x)$, *where* c *is any constant, is* $\mathbf{I}$*-integrable from* a *to* b, *and*

$$\mathbf{I}_a^b c\,F(x)\,dx = c\,\mathbf{I}_a^b F(x)\,dx.$$

* Theorem I follows as a corollary of theorem VI for *any* function $F(x)$ known to be I-integrable from a to b.

† The content of a point-set lying on an infinite interval is defined in a footnote of §1 13°.

‡ Even if ab is an infinite interval.

VII. *If the functions $F(x)$, $G(x)$ are $\mathbf{I}$-integrable from a to b, so is the function $F(x)+G(x)$, and*

$$\mathbf{I}_a^b\big(F(x)+G(x)\big)\,dx=\mathbf{I}_a^b F(x)\,dx+\mathbf{I}_a^b G(x)\,dx.$$

VIII. *The transformation of definite integrals.*

From the hypotheses:

1) *The x-interval ab and an $\bar{x}$-interval $\bar{a}\bar{b}$ (each interval being finite or infinite) are set in one-to-one point correspondence with preservation of sense by the transformation*

$$x=\phi(\bar{x})\ ^{*} \qquad (\bar{x}\text{ on }\bar{a}\bar{b}), \tag{3}$$

where

$$a=\phi(\bar{a}),\quad b=\phi(\bar{b}); \tag{4}$$

2) *The x-set Ξ_{ab} and an $\bar{x}$-set $\bar{\Xi}_{\bar{a}\bar{b}}$ correspond under the transformation* (3);

3) *On the $\bar{x}$-interval $\bar{a}\bar{b}$ apart from points $\bar{\xi}$ and limit-points $\bar{\xi}'$ of $\bar{\Xi}_{\bar{a}\bar{b}}$ the function $\phi(\bar{x})$ has a continuous finite and non-zero derivative† $\phi'(\bar{x})$, follow the conclusions:*

1) *Under the transformation* (3) *the type $\mathbf{I}$ on the interval ab with respect to the set Ξ_{ab} transforms into a type $\mathbf{I}$ on the interval $\bar{a}\bar{b}$ with respect to the set $\bar{\Xi}_{\bar{a}\bar{b}}$;*

2) *The function*

$$\bar{F}(\bar{x})\equiv F\big(\phi(\bar{x})\big)\phi'(\bar{x}) \tag{5}$$

is $\mathbf{I}$-integrable from $\bar{a}$ to $\bar{b}$;

3) *The formula*

$$\mathbf{I}_a^b F(x)\,dx=\bar{\mathbf{I}}_{\bar{a}}^{\bar{b}}\bar{F}(\bar{x})\,d\bar{x} \tag{6}$$

of transformation of definite integrals has validity.

3°. These theorems, with the exception of the second, are valid for proper definite integrals in the usual sense; they belong to the elements of the theory.

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* Where then $\phi(\bar{x})$ and its inverse $\bar{\phi}(x)$ are continuous functions of their respective arguments $\bar{x}$, x on the respective intervals $\bar{a}\bar{b}$, ab. (Cf. the second footnote of § 1 13°.) It is to be noted that a point-set Ω of content zero transforms under a transformation (3) subject to the condition 1) into a point-set $\bar{\Omega}$ likewise of content zero.

† Which has then the sign of $(b-a)/(\bar{b}-\bar{a})$ as permanent sign.—No statement is made concerning the existence or properties of the derivative $\phi'(\bar{x})$ at points $\bar{x}=\bar{\xi}$, $\bar{\xi}'$.—If the function $\phi(\bar{x})$ satisfies the conditions 1)–3), the inverse function $\bar{\phi}(x)$ satisfies the corresponding conditions on the x-interval ab. Similarly there is a composition of transformations of this character. Here one needs to *prove* (indirectly) that $\bar{a}$ or $\bar{b}$ belongs to $\bar{\Xi}+\bar{\Xi}'$ if it is infinite, so that, for example, no proper definite integral (over a finite interval) is (by theorem VIII) transformable into an integral over an infinite interval; this state of affairs accords with the general postulate of the close of § 1.

As to theorem VIII, cf. JORDAN'S *Cours d'Analyse*, ed. 2, vol. 1, nos. 141–144. The validity of all the theorems for proper integrals in the extended sense is easily perceived.

4°. *The* $\mathbf{I}$*-integrals over infinite intervals* ab *are defined* (cf. § 1 13°) *as transforms of* $\bar{\mathbf{I}}$*-integrals over finite intervals* $\bar{a}\,\bar{b}$ *by the definite transformation* $x = \psi(\bar{x})$ *of* § 1 13°.

That transformation $\left(\text{quâ } x = \phi(\bar{x})\right)$ satisfies the conditions of theorem VIII on the $\bar{x}$-interval* $-1 \leqq \bar{x} \leqq +1$.

By the notion of composition of transformations (3) one readily proves that the general validity of the theorems of § 2 follows from their validity for all finite intervals ab, $\bar{a}\,\bar{b}$ (the latter entering only in theorem VIII), and again that that validity follows from their validity for a single finite interval ab with all finite intervals $\bar{a}\,\bar{b}$.

§ 3.

FIRST DEFINITIONAL PROCESS OF INDUCTION.

1°. *By the first process of induction we are to define* (in 2°–4°) *from any set* Ξ_0 *and any type* $\mathbf{I}$ *of the set* Ξ *a* (*related*) *type* $\mathbf{I}_{\Xi_0}$ *of the aggregate set* $\Xi + \Xi_0$, that is, we are to characterize the functions $F(x)$ $\mathbf{I}_{\Xi_0}$-integrable from a to b. We agree at once that if $a = b$ every function $F(x)$ is so integrable, the integral having the value zero.

We assume the validity of theorems I–VIII *of* § 2 *for the type* $\mathbf{I}$, *and* (in 5°–7°) *we establish their validity for the type* $\mathbf{I}_{\Xi_0}$.

In accordance with § 2 4°, in these definitions and proofs we may and do restrict attention to the case of finite intervals ab.

In 8° I indicate a necessary and sufficient condition for the existence of the $\mathbf{I}_{\Xi_0}$-integral, which is very useful in the construction of integrals of type $\mathbf{I}_{\Xi_0}$ from those of type $\mathbf{I}$.

2°. The type $\mathbf{I}_{\Xi_0}$ depends upon the type $\mathbf{I}$ just as HARNACK'S general or narrow type† $\int_{(\Xi_0)}$ depends upon the type $\int$ of proper definite integrals, except that now the set Ξ_0 is of arbitrary content $\gamma(\Xi_0)$ and does not necessarily lie on a finite interval. As far as convenient I adopt the phrasings of my paper in the preceding number of this journal: On HARNACK'S Theory of Improper Definite Integrals,—and for brevity refer to it by the notation H. T.

3°. The following notations are needed.

(*a*) With respect to any interval ab and point-set Ξ or interval-set I the notation Ξ_{ab} or I_{ab} denotes the point-set of Ξ or interval-set lying on I which lies on ab.

* The extremities ± 1 are the transforms of $\pm\infty$ and so belong to the closed set $\bar{\Xi} + \bar{\Xi}'$.

† H. T. p. 304.

(β) When an interval-set * I encloses a point-set Ξ narrowly,* in which case it is denoted by the notation $I(\Xi)$, then its length D_I is greater than the content $\gamma(\Xi)$ of Ξ; this positive difference $D_I - \gamma(\Xi)$ we call the Ξ-length of I, and denote it by $D(I, \Xi)$; on the set of all such interval-sets $I(\Xi)$ the lower limit of $D(I, \Xi)$ is zero.

(γ) With respect to an interval-set I and a function $F(x)$ the function $F_I(x)$ is defined as having at every point x of I the value zero, and at every point x not of I the same determination as $F(x)$ at that point.

(δ) We denote the perfect set $\Xi_0 + \Xi_0'$ by Ξ^0.

4°. Our function $F(x)$ $\mathbf{I}_{\Xi_0}$-integrable from a to b (ab unequal and finite) is, in the first place, $\mathbf{I}$-integrable over every subinterval of ab containing no point ξ^0 of Ξ^0. This implies, according to theorems I, II, III, IV′ for the type $\mathbf{I}$, that $F_I(x)$ is $\mathbf{I}$-integrable from a to b, where I is any interval-set enclosing the point-set Ξ^0_{ab} narrowly.

Then in the second place, the (finite) limit:

$$\lim_{D(I,\,\Xi^0_{ab})=0}{}^{I \mid I(\Xi^0)} \mathbf{I}_a^b F_I(x)\,dx,$$

in notation †

$$\mathbf{I}_{a\,\Xi_0}^{\,b} F(x)\,dx,$$

exists; that is, for every positive ϵ a positive δ^1_ϵ ‡ exists such that

$$\left| \mathbf{I}_a^b F_I(x)\,dx - \mathbf{I}_{a\,\Xi_0}^{\,b} F(x)\,dx \right| < \epsilon$$

for every interval-set I enclosing Ξ^0_{ab} narrowly and of Ξ^0_{ab}-length less than δ^1_ϵ.—This limit is provably definite.—Thus in terms of the type $\mathbf{I}$ the type $\mathbf{I}_{\Xi_0}$ has been completely defined.

It is to be noticed that

$$\mathbf{I}_a^b F_I(x)\,dx = \mathbf{I}_{J_a^b} F(x)\,dx,$$

where J_a^b denotes the interval-set of ab complementary to I_{ab}, its intervals taken in the sense ab, and where the notation on the right denotes the sum of the $\mathbf{I}$-integrals of $F(x)$ over these directed intervals of J_a^b, or zero if J_a^b is non-existent. If in our definition we adopt $\mathbf{I}_{J_a^b} F(x)\,dx$ as the limitand expression, we have a form of definition closely analogous to that of JORDAN for the HARNACK-JORDAN integrals.

* H. T. § 1 5°, 7°.

† The notation $\mathbf{I}_{\Xi_0 a}^{\ b} F(x)\,dx$ would (except for its appearance) be preferable.

‡ In this paper superscripts to δ_ϵ notations are not exponents but mere affixes of discrimination.

5°. *For the type* $\mathbf{I}_{\Xi_0}$ *(quâ* $\mathbf{I}$*) the theorems* I–VIII *of* § 2 *hold.*—We admit the validity of these theorems for the type $\mathbf{I}$, and, as indicated in 1°, we suppose that a and b are finite, and furthermore distinct, since the theorems obviously hold if $a = b$. The truth of I and II for $\mathbf{I}_{\Xi_0}$ follows at once from their truth for $\mathbf{I}$. Theorems III, IV, IV′, V, V′, VI, VII for $\mathbf{I}_{\Xi_0}$ depend upon the lemmas given in 6°, essentially as the theorems H. T. § 3 V, VIII, § 2 III, § 3 IX, II, XVII depend upon the theorems H. T. § 3 IV, V, VI. Theorem VIII is considered in 7°.

6°. Lemma I. *For every positive* ϵ *there is a positive* δ^2_ϵ *such that*

$$\left| \mathbf{I}_{x_1}^{x_2} F_{I_1}(x)\,dx - \mathbf{I}_{x_1}^{x_2} F_{I_2}(x)\,dx \right| < \epsilon \tag{1}$$

for every two points $x_1 x_2$ *of the finite interval* ab *and every pair of interval-sets* $I_1 I_2$ *each enclosing* Ξ^0_{ab} *narrowly and of* Ξ^0_{ab}*-length less than* δ^2_ϵ.—Here the uniformity with respect to $x_1 x_2$ is of central importance.

This lemma corresponds to H. T. § 3 IV, the proof of which was involved with that of a more general theorem H. T. § 3 X, the general analogue of which is not now in question. I give as imple proof of the lemma.

The determination $\delta^2_\epsilon = \frac{1}{2}\delta^1_{\epsilon/2}$ is effective.

If the lemma is not true, then for a certain determination of ϵ, x_1, x_2, I_1, I_2 satisfying the specified conditions the inequality (1), call it (1′), is invalid. We may confine attention to the case $a < b$, $x_1 < x_2$.

Let the interval-set I enclose Ξ^0_{ab} narrowly and have Ξ^0_{ab}-length less than δ^2_ϵ. Let the interval $a\beta$ $(a < \beta)$ enclose ab $(a < b)$ so that

$$a < a \leqq x_1 < x_2 \leqq b < \beta.$$

Then each interval-set:

$$I_3 = I_{ax_1} + I_{1x_1x_2} + I_{x_2\beta},\quad I_4 = I_{ax_1} + I_{2x_1x_2} + I_{x_2\beta}, \tag{2}$$

encloses Ξ^0_{ab} narrowly and has* Ξ^0_{ab}-length less than $2\delta^2_\epsilon = \delta^1_{\epsilon/2}$.

Hence, obviously, recalling the definition, we have

$$\left| \mathbf{I}_a^b F_{I_3}(x)\,dx - \mathbf{I}_a^b F_{I_4}(x)\,dx \right| < \epsilon. \tag{3}$$

On the other hand, by (2) and § 2 IV, for the type $\mathbf{I}$, we have

$$\mathbf{I}_a^b F_{I_3}(x)\,dx = \mathbf{I}_a^{x_1} F_I(x)\,dx + \mathbf{I}_{x_1}^{x_2} F_{I_1}(x)\,dx + \mathbf{I}_{x_2}^b F_I(x)\,dx,$$

* We observe that I_3, I_4 are parts of $I_5 = I + I_1$, $I_6 = I + I_2$ respectively and use the following theorem of the theory of contents:

If the interval-sets I^1 *and* I^2 *enclose the point-sets* Ξ^1 *and* Ξ^2 *respectively, then the interval-set* $I = I^1 + I^2$ *encloses the point-set* $\Xi = \Xi^1 + \Xi^2$, *and*

$$D(I, \Xi) \leqq D(I^1, \Xi^1) + D(I^2, \Xi^2).$$

with a similar equation involving I_4 and I_2, so that

$$\mathbf{I}_a^b F_{I_3}(x)\,dx - \mathbf{I}_a^b F_{I_4}(x)\,dx = \mathbf{I}_{x_1}^{x_2} F_{I_1}(x)\,dx - \mathbf{I}_{x_1}^{x_2} F_{I_2}(x)\,dx,$$

whence by (1′) we have, in contradiction with (3),

$$\left|\mathbf{I}_a^b F_{I_3}(x)\,dx - \mathbf{I}_a^b F_{I_4}(x)\,dx\right| \geqq \epsilon. \tag{4}$$

Thus lemma I is proved.

Obviously lemma I holds also if the interval-sets $I_1 I_2$ enclose $\Xi^0_{x_1 x_2}$ (instead of Ξ^0_{ab}) narrowly, for the integrals in (1) are from x_1 to x_2. Hence by the usual limit-considerations theorem II is proved for the type $\mathbf{I}_{\Xi_0}$, and likewise

LEMMA II. *For every ϵ and every two points $x_1 x_2$ of the finite interval ab*

$$\left|\mathbf{I}_{x_1}^{x_2} F_I(x)\,dx - \mathbf{I}_{x_1 \Xi_0}^{x_2} F(x)\,dx\right| \leqq \epsilon$$

for every interval-set $\mathbf{I}$ enclosing Ξ^0_{ab} narrowly and of Ξ^0_{ab}-length less than δ^2_ϵ.

Lemma II is of fundamental importance in the deduction of theorems of linear form for the type $\mathbf{I}_{\Xi_0}$ from similar theorems for the type $\mathbf{I}$.

7°. The truth of theorem VIII (with ab, $\bar{a}\bar{b}$ finite intervals) for the type $\mathbf{I}_{\Xi_0}$(quâ $\mathbf{I}$) of the set $\Xi + \Xi_0$(quâ Ξ) follows easily from its truth for the type $\mathbf{I}$ of the set Ξ.

We consider a transformation

$$x = \phi(\bar{x}) \qquad (\bar{x} \text{ on } \bar{a}\bar{b}) \tag{5}$$

which satisfies the conditions of theorem VIII for the type $\mathbf{I}_{\Xi_0}$ of the set $\Xi + \Xi_0$.

Using notations of evident meaning we observe that the sets $\Xi_0 \Xi^0$ correspond to similarly related sets $\bar{\Xi}_0 \bar{\Xi}^0$; that every $\bar{I}$ enclosing $\bar{\Xi}^0$ narrowly corresponds to an I enclosing Ξ^0 narrowly; that in the notation of 4° every $\bar{J}^{\bar{b}}_{\bar{a}}$ corresponds to a J^b_a; and finally we observe, by considerations of the theory of contents of linear point-sets, in view of the continuity of the transformation (5) between $ab\ \bar{a}\bar{b}$, that for a positive ϵ a positive δ^1_ϵ exists such that the Ξ^0_{ab}-length of I is less than δ^1_ϵ in case the $\bar{\Xi}^0_{\bar{a}\bar{b}}$-length of $\bar{I}$ is less than δ^1_ϵ.

Then

$$\left|\mathbf{I}_{J_b^a} F(x)\,dx - \mathbf{I}_{a\,\Xi_0}^{b} F(x)\,dx\right| < \epsilon \tag{6}$$

for every interval-set $\bar{I}$ enclosing $\bar{\Xi}^0_{\bar{a}\bar{b}}$ narrowly and of $\bar{\Xi}^0_{ab}$-length less than δ^1_ϵ; and simultaneously, for every (directed) interval j of J^b_a,

$$\mathbf{I}_j F(x)\,dx = \mathbf{I}_{\bar{j}} \bar{F}(\bar{x})\,d\bar{x}, \tag{7}$$

since the transformation (5) on $j\bar{j}$ satisfies the conditions of theorem VIII for the type $\mathbf{I}$ of the set Ξ; and thus

$$\mathbf{I}_{J_a^b} F(x)\,dx = \overline{\mathbf{I}}_{\bar{J}_{\bar{a}}^{\bar{b}}} \bar{F}(\bar{x})\,d\bar{x}. \tag{8}$$

From (6) and (8) by elimination we have the inequality needed to prove the validity of theorem VIII for the type $\mathbf{I}_{\Xi_0}$.

8°. The theorem of H. T. § 5 2° may readily be generalized to the case of $\mathbf{I}_{\Xi_0}$-integrals. In the proof of the generalized theorem one uses instead of the theorem H. T. § 3 XIII′ considerations of continuity and definitional considerations concerning the analogues of the integrals, loc. cit. (3). This theorem is of fundamental use in the construction (as in H. T. § 5) from the functions $\psi_{01}(x)$, $\chi_{01}(x)$ of H. T. § 5 4° of functions having integrals of type $\mathbf{I}$ of a set Ξ partitioned into a finite sequence $\Xi_1, \Xi_2, \cdots, \Xi_m$, viz.

$$\mathbf{I} \equiv \int_{\Xi_1 \Xi_2 \cdots \Xi_m}.$$

§ 4.

The second definitional process of induction.

1°. In the second process of induction we have to do with a simply ordered set* $M = \{m\}$ of symbols or marks m and with a related simply ordered set $\{\mathbf{I}^m\}$ of types $\mathbf{I}^m$ of the point-set Ξ^m, where Ξ^{m_1} is a proper subset of Ξ^{m_2} if $m_1 < m_2$.

We are to define in 4° *a type* $\mathbf{I}^M$ *of the aggregate set* Ξ^M, *where* Ξ^M *denotes the* (*least*) *point-set containing every set* Ξ^m *as a subset.*

From this definition, on the assumption of the validity of theorems I–VIII *of* § 2 *for the types* $\mathbf{I}^m$, *follows their validity for the type* $\mathbf{I}^M$, as one readily assures himself.

2°. The simply ordered set $M = \{m\}$ may have a highest mark m, $m = m'$; in this case we say that it terminates with the mark m'. Every finite set M terminates. A transfinite set M may or may not terminate.

3°. Even if the set $\{\Xi^m\}$ does not terminate the set $\{\Xi^m\}$ may on an interval i essentially terminate, i. e., there may be a mark m_0 such that $\Xi_i^m \equiv \Xi_i^{m_0}$ for every $m > m_0$, where Ξ_i^m denotes the subset of Ξ^m lying on i.

* With Cantor (Mathematische Annalen, vol. 46, p. 496) a set M of marks m is simply ordered if of every two distinct marks $m_1 m_2$ one, say m_1, is distinguished, and if this distinction is transitive, i. e., if of the marks $m_1 m_2$ the mark m_1 is distinguished, and of $m_2 m_3$ m_2 is distinguished, then of the marks $m_1 m_3$ the mark m_1 is distinguished. One says that according to the ordering in question m_1 has the lower and m_2 the higher rank, in notation:

$$m_1 < m_2, \quad m_2 > m_1.$$

A point $x = x_0$ is with respect to the non-terminating transfinite set $\{\Xi^m\}$ critical, in case on no interval i enclosing x_0 the set $\{\Xi^m\}$ is essentially a terminating set. We denote these critical points x by the notation κ and the set of all such points by the notation K. The set K is closed.

If an interval i contains no critical point, the set $\{\Xi^m\}$ on i essentially terminates, as one proves by the usual interval-halving process.

Of course, the set K exists only if $\{\Xi^m\}$ is non-terminating; and even then it may contain no finite point.

4°. We are to define an (existent) integral

$$\mathbf{I}_a^{bM} F(x)\,dx \tag{1}$$

of the type $\mathbf{I}^M$, that is, to characterize the function $F(x)$ in order that the symbol (1) may have a (definite finite) value.

If $b = a$, we give to the symbol (1) the value 0.

If on the interval ab the function $F(x)$ vanishes except perhaps at a set of points of content zero, we give to the symbol (1) the value 0.

Further, if the interval ab contains no critical point κ and so on ab the set $\{\Xi^m\}$ is essentially terminating, and if there is a mark m_0 such that for all marks m, $m \geqq m_0$, the $\mathbf{I}^m$-integrals of $F(x)$ from a to b exist, and for every subinterval $a'b'$ of ab and mark $m > m_0$

$$\mathbf{I}_{a'}^{b'm} F(x)\,dx = \mathbf{I}_{a'}^{b'm_0} F(x)\,dx, \tag{2}$$

then we denote by $\mathbf{I}_a^{bM} F(x)\,dx$ the common value of the existent integrals

$$\mathbf{I}_a^{bm} F(x)\,dx \qquad (m \geqq m_0).$$

Finally, if the finite or infinite interval ab splits into a finite number n of intervals $a_k b_k (k = 1, \cdots, n)$ taken in the sense of ab in such a way that by the specifications already made the integrals $\mathbf{I}_{a_k}^{b_k M} F(x)\,dx$ are defined, we set

$$\mathbf{I}_a^{bM} F(x)\,dx \equiv \sum_{k=1}^{n} \mathbf{I}_{a_k}^{b_k M} F(x)\,dx. \tag{3}$$

Thus the symbol (1) is given, if any, in fact a definite finite value.

It is clear that if the set M terminates, with m say, then the types $\mathbf{I}^M$ $\mathbf{I}^m$ are identical.

§ 5.

The definition of the general type $\mathbf{I}$ of the set Ξ.

1°. With Cantor* a well ordered set $W = \{w\}$ of marks w is a simply ordered set with the property that its every subset (including the set itself) has a mark

* Mathematische Annalen, vol. 49 (1897), pp. 207, 208.

of lowest rank. Every subset of W is itself a well ordered set. I denote by $W_1 = \{w_1\}$ the set of all marks $w = w_1$ having immediately preceding marks $w = w_1^0$; the mark of lowest rank by w_0; and the residual set by $W_2 = \{w_2\}$. It is to be noticed that the marks $w = w_1^0$ include all the marks of W except the terminal mark, if there is one.

2°. For the definition of the general type $\mathbf{I}$ of improper integrals with respect to the point-set Ξ I consider the general well ordered set W such that Ξ admits an exhibition, in the notation of §4 1°, of the form $\Xi \equiv \Xi^W$, and I consider the general exhibition of Ξ in this form. The understanding is, however, that $\Xi^{w_0} \equiv \{\pm\infty\}$ and that for every w_2 the set Ξ^{w_2} is the least set containing as subsets the sets Ξ^w, $w \prec w_2$.

We set $\Xi_{w_1} \equiv \Xi^{w_1} - \Xi^{w_1^0}$; thus Ξ_{w_1} denotes an existing point-set.

3°. From the type $\mathbf{I}^{w_1^0}$ of the set $\Xi^{w_1^0}$ with the properties I–VIII of § 2 and the set Ξ_{w_1} we can, by the first process of induction, define the type $\mathbf{I}^{w_1}$ of the set Ξ^{w_1} with the same properties.

Likewise from the types $\mathbf{I}^w$, $w \prec w_2$, of the respective sets Ξ^w we can define the type $\mathbf{I}^{w_2}$ of the set Ξ^{w_2} by the second process of induction.

Likewise from all the types $\mathbf{I}^w$, w of W, of the respective sets Ξ^w we can define the type $\mathbf{I}^W$ of the set $\Xi^W \equiv \Xi$ by the second process.

4°. Now we take as type $\mathbf{I}^{w_0}$ of the set $\Xi^{w_0} \equiv \{\pm\infty\}$ on finite intervals the (extended) type of proper definite integrals defined in § 1 16°. From this type follow the definitions of all the types $\mathbf{I}^w$ and of the type $\mathbf{I}^W$. For, if not, there is an undefined type $\mathbf{I}^{w'}$ of lowest rank w'; and this is impossible, for, by the statements just made, w' is neither w_0 nor a w_1 nor a w_2.—*Thus the general type* $\mathbf{I}$ *of the set* Ξ *has definition uniquely related to the general exhibition of* Ξ *in the specified partitional form* $\Xi \equiv \Xi^W$, *where* W *denotes a well ordered set and where similar well ordered sets* W *are for the purposes of the definition equivalent.*

5°. In conclusion, we point out how the system of types of improper integrals thus defined embraces the four current types as extended (§1 1°, 16°).

TYPE I.—In the first type the set Ξ is reducible, that is, for some (lowest) ordinal $a \geqq 1$ of CANTOR'S first or second class of ordinals the derived set $\Xi^{(a)} \equiv 0$. We consider the set $\{\beta\}$ of ordinals β, $1 \leqq \beta < a$, arranged in their natural ascending order. This set $\{\beta\}$ is a well ordered set. We consider the similar well ordered set $\{w_{1\beta}\}$ and make it the set W_1 of a well ordered set W, $W = w_0 + W_1 + W_2$, by introducing the initial mark w_0 and a set W_2 of marks $w_2 = w_{2\beta_2}$; here β_2 denotes an ordinal > 1 having no immediately preceding ordinal, while in well ordered set W the mark $w_{2\beta_2}$ immediately precedes the mark $w_{1\beta_2}$. Then the set Ξ^{w_0} is $\{\pm\infty\}$, the set $\Xi^{w_{1\beta}}$ contains all

points ξ not belonging to the derived set $\Xi^{(\beta)}$, and the set $\Xi^{w_2\beta_2}$ is the least set containing as subsets the sets $\Xi^{w_1\beta}$, $\beta < \beta_2$, and so the least set containing as subsets the sets Ξ^{w}, $w \prec w_{2\beta_2}$. The set $\Xi^{w'}$ is a proper subset of $\Xi^{w''}$ if $w' \prec w''$.

TYPE II.—In the second type the set Ξ is any set containing $\pm\infty$; the set W consists of two ordered elements w_0, w_1; and $\Xi^{w_0} = \{\pm\infty\}$, $\Xi^{w_1} = \Xi$.

TYPE III.—The third type is essentially the same as the first type.

TYPE IV.—The fourth type is in a sense (sufficiently indicated in § 1 9°) a combination of the second and the first types.

THE UNIVERSITY OF CHICAGO,
October 2, 1901.

ON THE CONVERGENCE AND CHARACTER OF THE CONTINUED FRACTION

$$\frac{a_1 z}{1} + \frac{a_2 z}{1} + \frac{a_3 z}{1} + \cdots.$$*

BY

EDWARD B. VAN VLECK

The principal object of the present paper is to give two theorems (nos. I and IV) concerning the character of the function represented by the continued fraction

$$\frac{a_1 z}{1} + \frac{a_2 z}{1} + \frac{a_3 z}{1} + \cdots.$$

Incidentally a criterion for the convergence of

$$\frac{b_1}{1} + \frac{b_2}{1} + \frac{b_3}{1} \cdots$$

is obtained, which appears to be more general than any hitherto discovered except for the special case in which the elements b_n are positive. This is stated in theorem III.

§ 1.

The first part of the paper relates to the following theorem:

THEOREM I. *If k denotes the greatest modulus of a point of condensation of the coefficients a_n of the continued fraction*

$$\frac{a_1 z}{1} + \frac{a_2 z}{1} + \frac{a_3 z}{1} + \cdots, \tag{1}$$

the continued fraction will represent an analytic function within a circle of radius $1/4k$, described about the origin of the z-plane as center, and the only singularities of the function contained within the circle will be poles.

* Presented to the Society August 20, 1901, at its summer meeting as a part of the paper entitled *On the Convergence of the Continued Fraction of Gauss.* Received for publication September 23, 1901. The remainder of the paper is published in the Annals of Mathematics, October, 1901.

If, furthermore, any circle of smaller radius is drawn about the origin as center and from this circle each of the poles is excluded by drawing around it a small but arbitrary contour, then within the region which remains the continued fraction will converge uniformly to the analytic function as its limit.

A special case of this theorem is that in which $|a_n|$ diminishes indefinitely with increasing n. The continued fraction then represents a function which is meromorphic over the entire finite plane. This particular result, though in a yet more restricted form, has been previously discovered by STIELTJES.* Confining his attention to a continued fraction in which a_n is real and positive, he proved that it represents a meromorphic function when $\lim. a_n = 0$, and only then.

§ 2.

The proof of the above theorem is as follows: Let C be a circle whose center is the origin and whose radius is $e/4k$, in which e is taken to denote some fixed number less than unity. If e_1 is any other assigned number greater than e but less than unity, the value of $|a_n z|$ throughout the circle—inclusive of the perimeter—may be made less than $e_1/4$ by sufficiently increasing n. Let this be true when $n > \mu$, and consider the continued fraction

$$\frac{1}{1} + \frac{a_{\mu+1}z}{1} + \frac{a_{\mu+2}z}{1} + \cdots. \tag{2}$$

We shall first demonstrate the convergence of (2). Let N_n and D_n be the numerator and denominator of its nth convergent. Obviously $2|D_2| > |D_1|$. Suppose also that

$$2|D_n| > |D_{n-1}| \tag{3}$$

for any particular value of n. Since

$$D_{n+1} = D_n + a_{n+\mu} z D_{n-1}, \tag{4}$$

it follows that

$$|D_{n+1}| > |D_n|\left(1 - \frac{e_1}{2}\right),$$

and hence $2|D_{n+1}| > |D_n|$. The inequality (3) therefore holds for all values of n and accordingly

$$\frac{|D_{n+1}|}{|D_{n-1}|} > \tfrac{1}{4}.$$

A similar result holds for the numerators of the convergents.

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*Annales de la Faculté des Sciences de Toulouse, vol. 9, *A* pp. 42-47.

Consider next the equation

$$\frac{N_{n+r+1}}{D_{n+r+1}} - \frac{N_n}{D_n} = (-1)^n a_{\mu+1}a_{\mu+2}\cdots a_{\mu+n}z^n\left(\frac{1}{D_nD_{n+1}} - \frac{a_{\mu+n+1}z}{D_{n+1}D_{n+2}}\right.$$

$$\left. + \frac{a_{\mu+n+1}a_{\mu+n+2}z^2}{D_{n+2}D_{n+3}} - \cdots + (-1)^r \frac{a_{\mu+n+1}a_{\mu+n+2}\cdots a_{\mu+n+r}z^r}{D_{n+r}D_{n+r+1}}\right). \tag{5}$$

Since $|a_{\mu+i}z| < e_1/4$ and $4|D_{i+1}D_i| > |D_iD_{i-1}|$, the modulus of the right hand member is less than

$$\frac{e_1^n}{4^n|D_nD_{n+1}|}(1 + e_1 + e_1^2 + \cdots + e_1^r),$$

and hence less than

$$\frac{e_1^n}{4(1-e_1)|D_1D_2|}.$$

But by increasing n the last quantity may be made as small as desired. If, therefore, an arbitrarily small quantity ϵ is prescribed, a value m can be found such that

$$\left|\frac{N_{n+r+1}}{D_{n+r+1}} - \frac{N_n}{D_n}\right| < \epsilon \qquad (n > m).$$

It follows that N_{n+r+1}/D_{n+r+1} converges uniformly to a limit in the circle C. The convergence of the continued fraction (2) has accordingly been established.

The character of the limiting function can next be easily determined. The inequality (3) and the similar inequality connecting N_n and N_{n-1} show, in the first place, that neither the numerator nor the denominator of the nth convergent ever vanishes. The convergents are, therefore, holomorphic throughout C. Since also they converge uniformly to a limit, their limit must be holomorphic in the interior of C. Furthermore, whenever a series of holomorphic functions converge uniformly to a limit, the zeros of the limiting function are the condensation points of the zeros of the rational functions.* We conclude, therefore, that the limiting function cannot vanish within C.

A case of special interest is that in which $|a_{\mu+n}| \leqq k$ for all values of n in the continued fraction (2). The radius of C may then be taken as nearly equal to $1/4k$ as desired, and we have in consequence the following result:

THEOREM II. *If in any continued fraction*

$$\frac{1}{1} + \frac{b_2z}{1} + \frac{b_3z}{1} + \cdots$$

$|b_n|$ is equal to or less than a fixed quantity k for all values of n, the continued fraction converges within a circle of radius $1/4k$ described about the origin of the

* HURWITZ, Mathematische Annalen, vol. 33.

z-plane as center, and it represents a function which is holomorphic within the circle and vanishes nowhere in its interior.

§ 3.

We return now to our continued fraction (1). If the mth convergent is denoted by N'_m/D'_m, we have

$$\frac{N'_{\mu+n-1}}{D'_{\mu+n-1}} = \frac{N'_{\mu-1} + a_\mu z \frac{N_n}{D_n} N'_{\mu-2}}{D'_{\mu-1} + a_\mu z \frac{N_n}{D_n} D'_{\mu-2}}. \tag{6}$$

But in § 2 it was shown that N_n/D_n converges uniformly to a function—call it $\phi(z)$—which is holomorphic within C. Hence the numerator and denominator of the right hand member of (6) converge uniformly, and their quotient approaches the limit

$$\frac{N'_{\mu-1} + a_\mu z N'_{\mu-2}\phi(z)}{D'_{\mu-1} + a_\mu z D'_{\mu-2}\phi(z)}.$$

It is impossible that the denominator of this expression should vanish identically, for we have

$$\frac{D'_{\mu-2}}{D'_{\mu-1}} = \frac{1}{1} + \frac{a_{\mu-1}z}{1} + \frac{a_{\mu-2}z}{1} + \cdots + \frac{a_2 z}{1},$$

which would give a second continued fraction for $\phi(z)$ inconsistent with (2).* The limit is therefore an analytic function.

Furthermore, since the numerator and denominator of the above expression are holomorphic in C, its only singularities will be poles. Let each of these poles be excluded from C by drawing a circle of small but finite radius around it. We then have left a region within which not only the numerator and denominator of the right hand member of (6) converge, but also their quotient, $N'_{\mu+n-1}/D'_{\mu+n-1}$. The convergence of the continued fraction is thus established. Since also by increasing μ the radius of C may be made as nearly equal to $1/4k$ as we please, the theorem enunciated at the beginning of the paper follows.

It will be observed that the proof involved the fact that $N'_{\mu-1}$, $N'_{\mu-2}$, $D'_{\mu-1}$, $D'_{\mu-2}$ are polynomials, and did not depend upon the monomial form of the numerators and denominators of the first $\mu - 1$ partial quotients. The theorem will therefore hold if a finite number of partial quotients are replaced by arbitrarily selected rational functions.

§ 4.

No information is afforded in theorem I concerning the character of the continued fraction without the circle considered. This question is discussed for

*See PADE'S thesis, Annales de l'Ecole Normale, sér. III, vol. 9 (1892), supplément.

an important case in the Annals of Mathematics, October, 1901, namely, for the case in which a_n approaches a finite limit k in such a manner that a_{2n-1}/a_{2n} and a_{2n}/a_{2n+1} for sufficiently great values of n can be expanded into series in ascending powers of $1/n$. It is shown that *the domain within which the function represented by the continued fraction is meromorphic can then be extended over the entire plane with the exception of a rectilinear cut from* $x = -1/4k$ *to* $x = \infty$, *drawn in a direction which is the continuation of the line from the origin to* $x = -1/4k$. Application of this result is made to the continued fraction of GAUSS.

§ 5.

Theorem II gave a sufficient condition that the limiting function shall be holomorphic within the circle in question. This condition is, however, by no means a necessary one. To obtain a more general condition that there shall be no poles of the function within the circle, consider the continued fraction

$$\frac{1}{1+}\ \frac{r_2 e^{i\theta_2}}{1}\ +\ \frac{r_3(1-r_2)e^{i\theta_3}}{1}\ +\ \frac{r_4(1-r_3)e^{i\theta_4}}{1}\ +\ \cdots, \tag{7}$$

in which r_n is a positive number less than 1. Obviously

$$|D_2| \geqq (1-r_2)\,|D_1| = 1 - r_2.$$

If, now, for any value of n

$$|D_n| \geqq (1-r_n)|D_{n-1}| \geqq (1-r_2)(1-r_3)\cdots(1-r_n), \tag{8}$$

we shall have also

$$|D_{n+1}| \equiv |D_n + r_{n+1}(1-r_n)e^{i\theta_n}D_{n-1}| \geqq |D_n| - r_{n+1}(1-r_n)\,|D_{n-1}|,$$

and hence

$$|D_{n+1}| \geqq |D_n|(1-r_{n+1}) \geqq (1-r_2)(1-r_3)\cdots(1-r_{n+1}).$$

The inequality (8) therefore holds for all values of n.

By a well known formula, the nth convergent is the sum of n terms of the series

$$1 - \frac{r_2 e^{i\theta_2}}{D_2} + \frac{r_2 r_3(1-r_2)e^{i(\theta_2+\theta_3)}}{D_2 D_3} - \frac{r_2 r_3 r_4(1-r_2)(1-r_3)e^{i(\theta_2+\theta_3+\theta_4)}}{D_3 D_4} + \cdots. \tag{9}$$

In the particular case for which $\theta_n = \pi$ we have

$$D_n = (1-r_2)(1-r_3)\cdots(1-r_n),$$

and the ratio of the nth term of the series to the preceding is then $r_n/(1-r_n)$. When $r_n < 1$, this is clearly the case which is least favorable to the convergence of the continued fraction (7). The following result is now evident:

THEOREM III. *If r_n is a positive number, a necessary condition that the continued fraction*

$$\frac{1}{1}+\frac{r_2e^{i\theta_2}}{1}+\frac{r_3(1-r_2)e^{i\theta_3}}{1}+\frac{r_4(1-r_3)e^{i\theta_4}}{1}+\cdots$$

shall converge for all values of the arguments θ_n is that the series

$$(10)\qquad 1+\frac{r_2}{1-r_2}+\frac{r_2r_3}{(1-r_2)(1-r_3)}+\frac{r_2r_3r_4}{(1-r_2)(1-r_3)(1-r_4)}+\cdots$$

shall likewise converge. This is also a sufficient condition if $r_n<1$ for all values of n.

§ 6.

The continued fraction

$$(11)\qquad \frac{1}{1}+\frac{b_2}{1}+\frac{b_3}{1}+\cdots$$

is usually capable of being expressed in the form (7). Often we shall also have $r_n<1$. For example, this is true when

$$(12)\qquad |b_2|<\tfrac{1}{2},\quad |b_{2n+1}|+|b_{2n+2}|\leqq\tfrac{1}{2}\qquad (n>0).$$

For if we put

$$|b_2|=r_2,\quad |b_n|=|1-r_{n-1}|r_n\qquad (n>2),$$

and assume that $r_{2n}<\frac{1}{2}$, it follows from the second of the inequalities (12) that

$$(13)\qquad r_{2n+1}<1,\quad r_{2n+2}<\tfrac{1}{2}.$$

But $r_2<\frac{1}{2}$, and hence, by induction, the inequalities (13) hold for all values of n.

PRINGSHEIM * has shown that the conditions (12) suffice to ensure the convergence of a continued fraction of the type (11). This appears to be the most general criterion for convergence hitherto discovered except for the special case in which all the elements b_n are positive. It is clear, however, from (13) that PRINGSHEIM'S criterion is included as a special case under theorem III.

§ 7.

Assuming now r_n to be less than 1, we shall next consider the effect of multiplying a single partial numerator, $r_m(1-r_{m-1})e^{i\theta_m}$, by a quantity δ, the absolute value of which is less than 1. If we put $r'_m=|\delta_m|r_m$ and seek to preserve the form of the continued fraction unaltered, we must make

$$(1-r_m)r_{m+1}=(1-r'_m)r'_{m+1}.$$

*Sitzungsberichte der mathematisch-physikalischen Classe der Münchener Akademie, vol. 28, p. 322.

Consequently $r'_{m+1} < r_{m+1}$. In like manner it follows that $r'_{m+2} < r_{m+2}$, and so on. All the terms in (10) after the $(m-1)$th will therefore be diminished, the new values being less than the old values multiplied into $|\delta|$.

Let now δ be introduced into each numerator of (7). In accordance with what has just been said, the nth convergent must be less than the sum of n terms of the series

$$1 + |\delta| \frac{r_2}{1-r_2} + |\delta^2| \frac{r_2 r_3}{(1-r_2)(1-r_3)} + |\delta^3| \frac{r_2 r_3 r_4}{(1-r_2)(1-r_3)(1-r_4)} + \cdots.$$

Suppose that this converges for $|\delta| = \delta'$. Then if δ in the continued fraction is replaced by a variable z, the fraction will converge uniformly in a circle having its center in the origin of the z-plane and a radius equal to δ'. Its limit is accordingly holomorphic within the circle. Furthermore,

$$|D_n| > (1-\delta' r_2)(1-\delta' r_3)\cdots(1-\delta' r_n).$$

Hence it is impossible for the denominators of the convergents to vanish anywhere in the circle, and this is true also of the numerators, inasmuch as the law for their formation is similar to (4). Finally, by the theorem of HURWITZ previously cited, we conclude that the limiting function does not vanish. The result which we have thus reached may be recapitulated as

THEOREM IV. *If in the continued fraction*

$$\frac{1}{1} + \frac{r_2 e^{i\theta_2} z}{1} + \frac{r_3(1-r_2)e^{i\theta_3} z}{1} + \frac{r_4(1-r_3)e^{i\theta_4} z}{1} + \cdots$$

r_n is a positive number less than 1, the continued fraction will converge uniformly in a circle described about the origin of the z-plane with a radius equal to unity, provided the series (10) is convergent. If this series is divergent, the radius is at least equal to the largest positive value of δ for which

$$(14)\quad 1 + \delta \frac{r_2}{(1-r_2)} + \delta^2 \frac{r_2 r_3}{(1-r_2)(1-r_3)} + \delta^3 \frac{r_2 r_3 r_4}{(1-r_2)(1-r_3)(1-r_4)} + \cdots.$$

converges. The limit of the fraction is an analytic function which is holomorphic and nowhere vanishes within the circle, and the roots of the numerators and denominators of the convergents all lie without it.

An interesting application may be made to the well-known expansion

$$\tfrac{1}{2} x \log \frac{x+1}{x-1} = \frac{1}{1} - \frac{a_2 x^{-2}}{1} - \frac{a_3 x^{-3}}{1} - \cdots,$$

in which

$$a_{n+1} = \frac{n \cdot n}{(2n-1)(2n+1)}.$$

In this case

$$\theta_n = \pi, \quad r_n = \frac{n-1}{2n-1},$$

and (14) becomes

$$1 + \tfrac{1}{2}\delta + \tfrac{1}{3}\delta^2 + \tfrac{1}{4}\delta^3 + \cdots.$$

Now this series converges when $\delta < 1$. The continued fraction therefore possesses the properties indicated in the theorem over the portion of the plane which lies without a circle of unit radius with its center in the origin. On the other hand, when $\delta = 1$, the series diverges. As the sum of n terms for $\delta = 1$ and the value of the nth convergent for $x^{-2} = 1$ are identical, the continued fraction must diverge at the points $x = \pm 1$ upon the boundary of the circle.

The properties which this continued fraction possesses exterior to the circle hold also throughout the entire imaginary domain. This can be shown by throwing it into the form

$$\frac{x}{x} - \frac{1}{b_2 x} - \frac{1}{b_3 x} - \cdots,$$

in which

$$b_{2n} = \frac{1 \cdot a_3 \cdots a_{2n-1}}{a_2 a_4 \cdots a_{2n}}, \quad b_{2n+1} = \frac{a_2 a_4 \cdots a_{2n}}{a_3 a_6 \cdots a_{2n+1}},$$

Since b_n is positive and $\sum b_n$ is divergent, a theorem given on page 231 of the present volume of the Transactions can be applied. We conclude that, (1) the continued fraction converges everywhere except in the segment of the real axis between the points $x = \pm 1$, (2) the limit is holomorphic without this segment, and (3) the roots of the numerators and denominators of the convergents are contained within it. It will be noticed that the proof of this result here given, unlike previous demonstrations, is based *solely* upon the character of the continued fraction.

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NOTES AND ERRATA: VOLUMES 1 AND 2

VOLUME 1.

O. BOLZA: *The elliptic σ-functions* ···.

P. 54, l. 8 up. *For* J' *read* T'.

W. F. OSGOOD: *On the existence of the Green's function* ···.

Pp. 310–314. I desire to point out the relation of my paper "On the existence of the Green's function for the most general simply connected plane region" to the analysis contained in HARNACK's *Logarithm. Potential* (1887), § 39. HARNACK there proposes the problem of showing the existence of a Green's function corresponding to an arbitrary simply or multiply connected continuum, i. e., precisely the problem that I have solved for a simply connected continuum, the extension of my results to multiply connected continua being obvious. (The extension is, namely, this: A Green's function for a multiply connected continuum will always exist when the boundary of the region does not contain isolated points, but is such that with each point of the boundary may be associated two other points so chosen that the three points lie on a Jordan curve.) In the solution which follows he restricts himself to a simply connected continuum F bounded by a Jordan curve C (cf. footnote, p. 310 of my article) and by an arbitrary set of curves (Einschnitte), finite in number, which lie within C, meet C each in a single point, and do not cut themselves or each other. In order to solve the problem, he constructs a set of nested polygons lying within F and having the boundary points of F as their points of condensation. The Green's functions belonging to these polygons are shown to converge toward a limit g, corresponding to the function u of my paper, which is a function similar in character to the Green's functions just considered. Up to this point both HARNACK's methods and mine are substantially the same as those of POINCARÉ, Bulletin de la Société mathématique de France, vol. 11 (1883), p. 112; cf. also HARNACK's

reference to SCHWARZ, loc. cit., p. 121. It remains to show that the function g (or u) assumes the required boundary values. To do this HARNACK employs as a majorante the Green's function belonging to a polygon Q lying wholly without F and having a point of its boundary in common with a point A of the boundary of F. His analysis suffices to show that the function g (or u) will take on the required boundary value in the point A, but not that this will be the case for a point of the boundary of F that cannot be reached by a polygon Q. Thus an ordinary beak-shaped cusp (Schnabelspitze) could not be treated by HARNACK'S method. It appears, then, that HARNACK did not solve the problem he proposed even for regions F bounded by a finite number of pieces of analytic curves, to say nothing of regions, some of the points of whose boundaries cannot be approached along a continuous curve lying wholly within F. In my solution, I have employed the same method of the majorante (the function U) adopted by HARNACK, but have so chosen U that my proof covers *all* cases; and I have pointed out that there are here included cases which, I believe, had never been thought of before.—W. F. O.

P. 312, l. 1 up. *For* 167 *read* 67.
P. 314, l. 10. *After* whether *insert* if.

E. KASNER: *The invariant theory of the inversion group* ···.

P. 431, l. 6 up. The complete reference is: MAURER, *Ueber die Endlichkeit der Invarianten-Systeme*, Münchener Sitzungsberichte, vol. 29 (1899), pp. 147–175.
P. 440, l. 18. *For* $F(\lambda f + MQ)$ *read* $F_{\lambda f+MQ}$.
P. 443, l. 9. " $(ABCD)$ " $(ABCu)$.
P. 445, l. 12. The lower right hand element of the determinant g_{123} should be $\lambda_1\mu_1$.
P. 448, l. 17. *For* circles *read* cyclics.
P. 449, l. 3. " I_{ik} " I_{ik}^2.
P. 467, l. 13. " Σ' " Σ.
P. 469, l. 18. " x " ψ.
P. 469, l. 5 up. " ϕ " Φ.
P. 475, l. 15. " $a_1 - a_1$ " $a_4 - a_2$.
P. 477, l. 8 up. The expression in braces should be squared.
P. 480, l. 20. *For* l_1 *read* l.
P. 489, l. 5 up. " WEITER " WEILER.

E. H. MOORE: *A simple proof of the fundamental Cauchy-Goursat theorem.*

P. 501, l. 5. *For* $<\epsilon$ *read* $\leqq\epsilon$.
P. 504, ll. 3, 7, 11 up. " $<$ " $\leqq$.

VOLUME 2.

E. J. WILCZYNSKI: *Invariants of systems of linear differential equations.*

P. 9, l. 12. For semivariants *read* seminvariants.
P. 11, l. 17. " $y_k =$ " $\bar{y}_k =$.
P. 22, l. 3. Make the expression into an equation by the addition of $= 0$.

J. C. FIELDS: *On the reduction of the general Abelian integral.*

P. 79, l. 19. *For* $r + s + 2 = \sigma$ *read* $r + s - 2 = \sigma$.
P. 80, l. 2 up. " $n + 3$ " $n - 3$.
P. 85, l. 2. " $+\sum_{\lambda=1}^{d+p}$ " $-\sum_{\lambda=1}^{d+p}$.

H. F. STECKER: *On the determination of surfaces* ....

P. 155, l. 11 up. Replace $d\mu$ in the expression for F_2 by $d\nu$.
" l. 3 up. The exponent of F_2 should be $-\frac{2}{3}$.
P. 159, l. 7. *For* $m + \beta - 1$ *read* $m - \beta - 1$.
P. 163, l. 17 up. " V_1 " V_2.
" l. 1 up. " $+\phi_2(\nu)$ " $-\phi_2(\nu)$.

E. B. VAN VLECK: *On the convergence of continued fractions* ....

Pp. 223, 224. The last line of p. 224 is to be set at the top of p. 223.
P. 226, l. 9 up. *For* $-a_n M_{n-1}^2$ *read* $a_n M_{n-1}^2$.
P. 233, l. 16. " $|a_n|/|\beta_n|$ " $a_n/|\beta_n|$.

W. F. OSGOOD: *On a fundamental property of a minimum* ....

P. 293, l. 7. *For* its longest side *read* the greatest of the differences $\tau_{i+1} - \tau_i$.

E. H. MOORE: *Concerning Harnack's theory* ···.

P. 296, l. 4 up. *For* DE LA VALLÉE POUSSIN *read* DE LA VALLÉE-POUSSIN.
P. 302, l. 19 up. " otherwise " , otherwise, .
P. 304, l. 2. *After* there *insert* exists.
P. 305, l. 10. In formula (8) insert an upper limit b.
P. 309, l. 2. } Clearly in this connection Z_{ab} should contain a or b only
P. 311, l. 8. } in case a or b is singular towards the inside of the interval ab.
P. 309, l. 14. *For* $\delta^1_{\epsilon/2}$ *read* $\delta_{\epsilon/2}$.
P. 319, l. 13 up. " J " J,.
P. 324, l. 10. " every " every ν.
P. 329, l. 15. " ν " ν'.